From tiling problems to random matrices

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Brussels Summer School of Mathematics

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Outline

Tiling problems

1. Some fun to start with
2. Large random tilings

Non-intersecting random walks and Brownian bridges

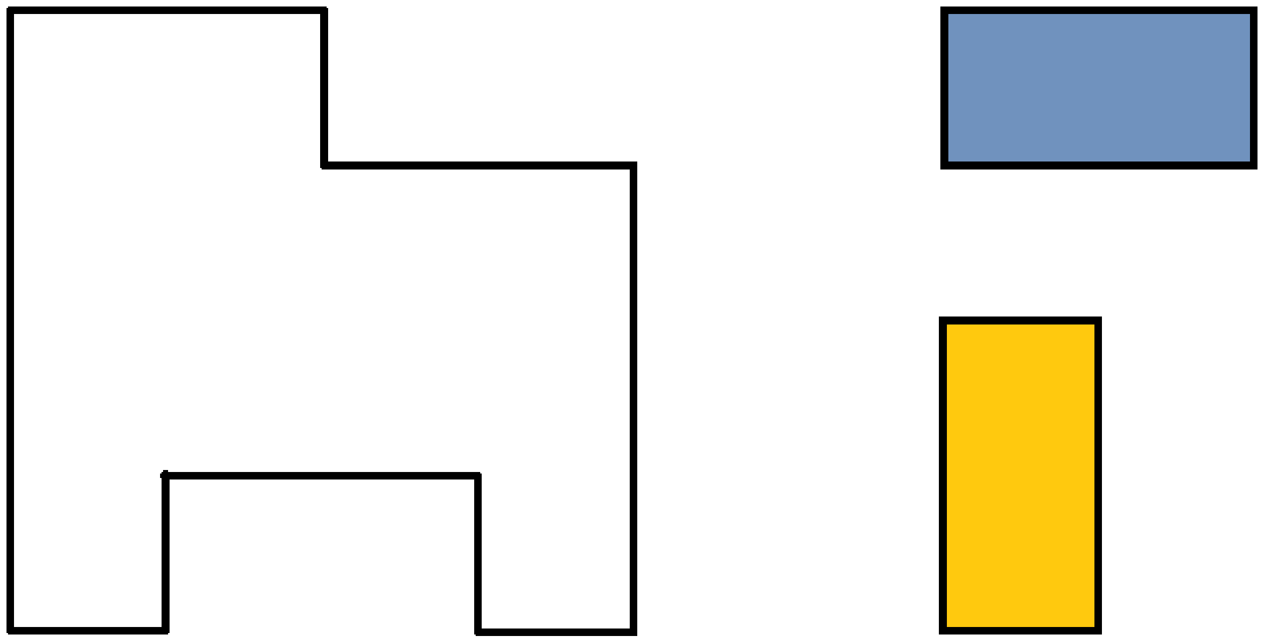
1. From hexagon tilings to non-intersecting paths
2. From non-intersecting paths to Brownian bridges

Random matrices

1. From non-intersecting Brownian bridges to random matrix eigenvalues
2. Asymptotic properties of random matrix eigenvalues

Tiling problems

Given a two-dimensional domain and a collection of (shapes of) tiles, we try to cover the domain with the tiles.



Tiling problems

Rules

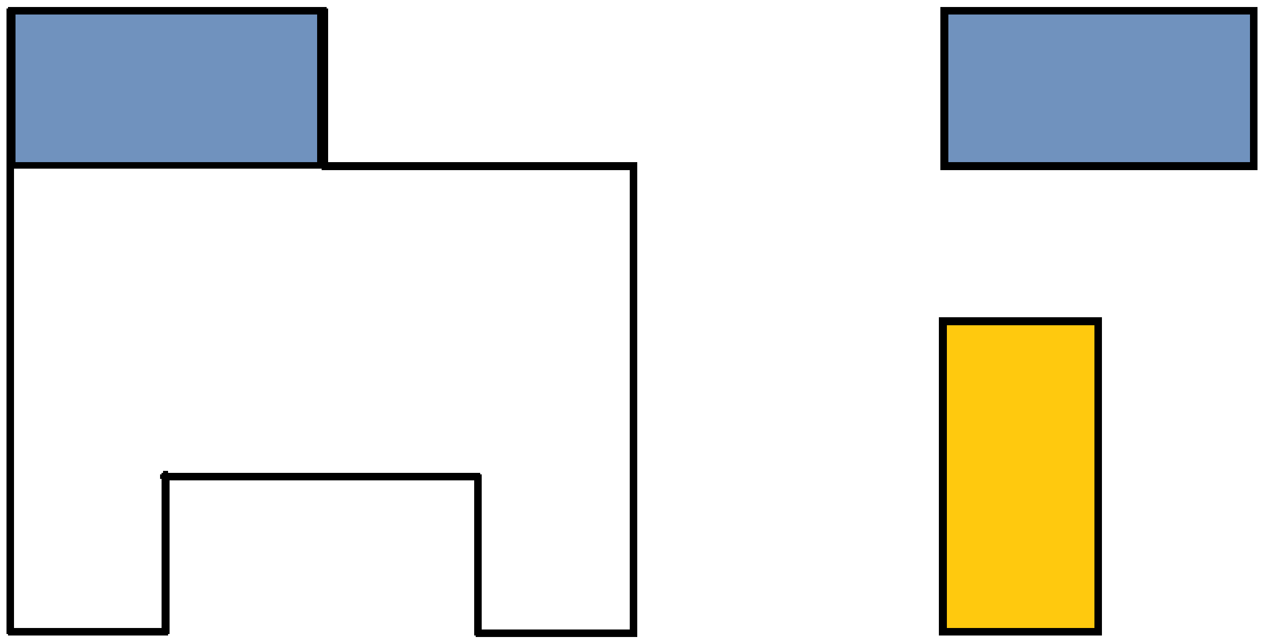
- ✓ **Allowed:** translations of tiles
- ✓ **Forbidden:** rotations, intersections, cutting tiles, crossing the border of the domain

Questions

- ✓ **Solvability?** Can the domain be covered with tiles?
- ✓ What is the **number of possible tilings**?
- ✓ Do different tilings share certain properties?

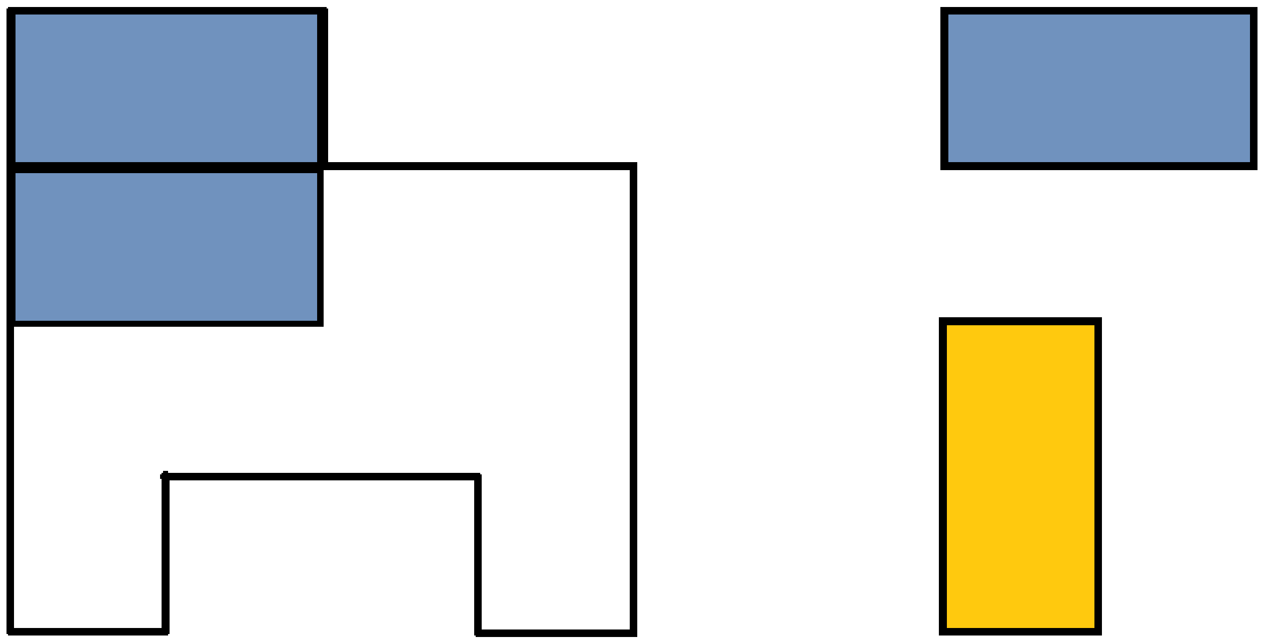
Tiling problems

A possible tiling



Tiling problems

A possible tiling



Tiling problems

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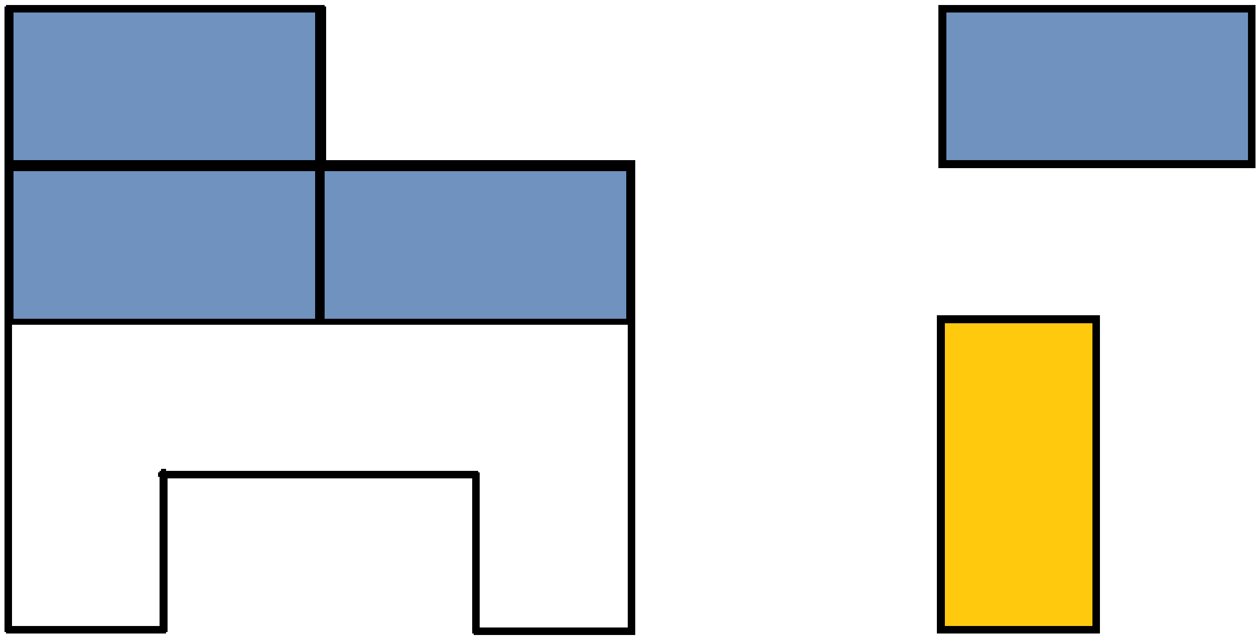

Tiling problems

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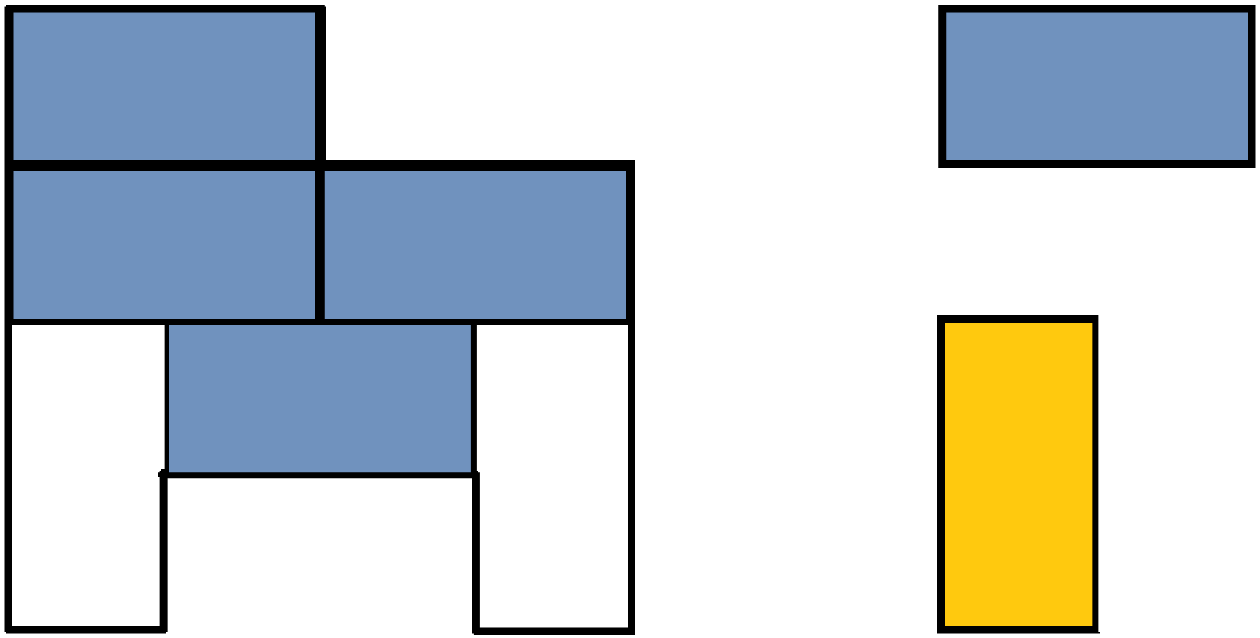

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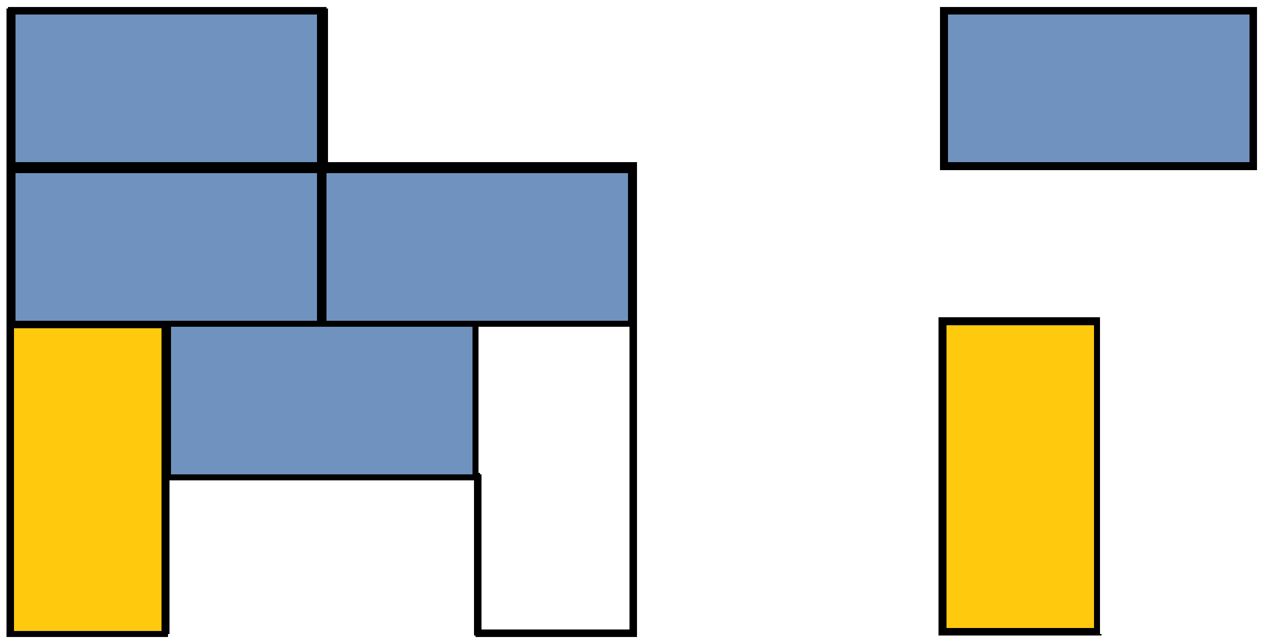

Tiling problems

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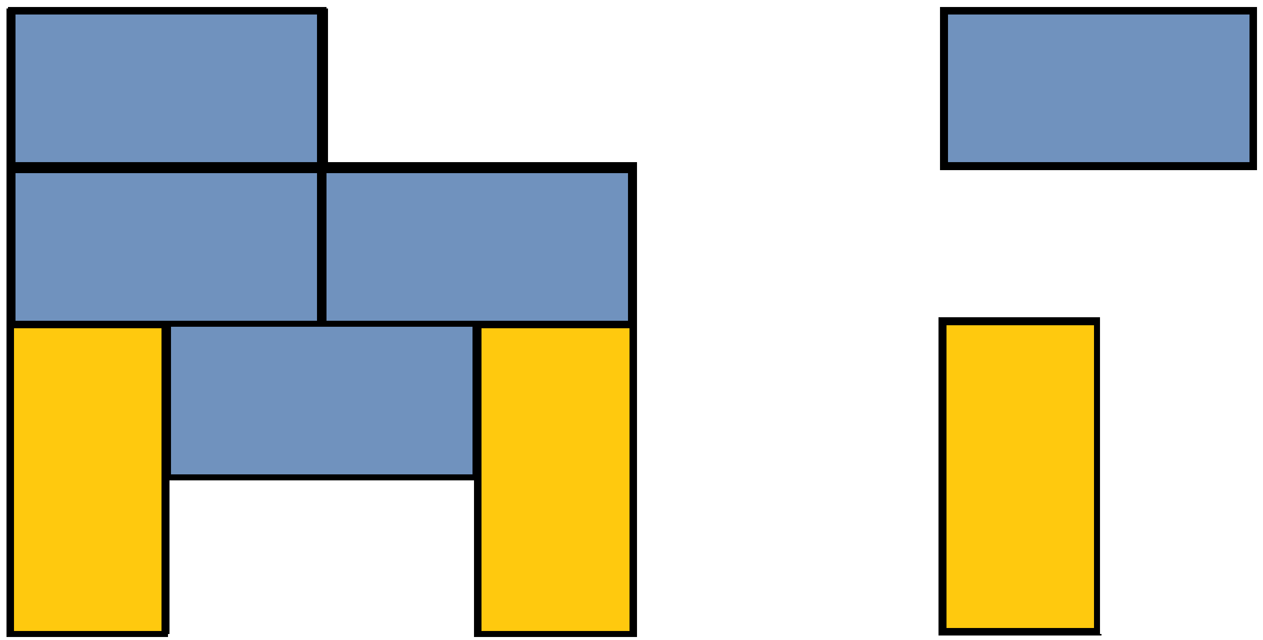
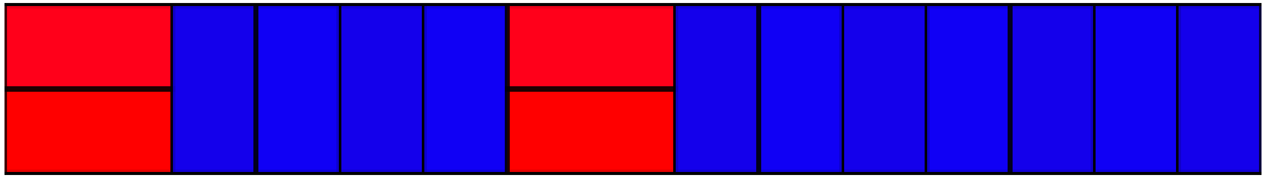


Tilings of a $2 \times n$ rectangle

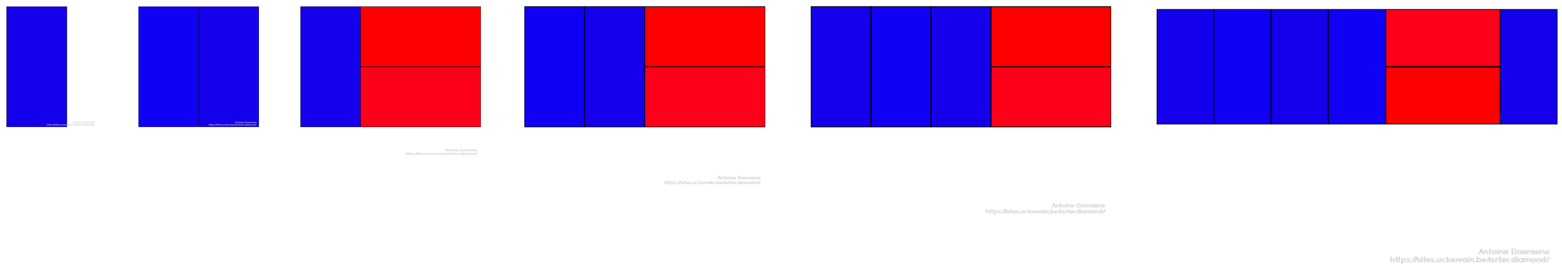
We speak of a **domino tiling** if the tiles or 1×2 and 2×1 rectangles.

First training example: tiling of a rectangle of height 2



Tilings of a $2 \times n$ rectangle

Number of tilings



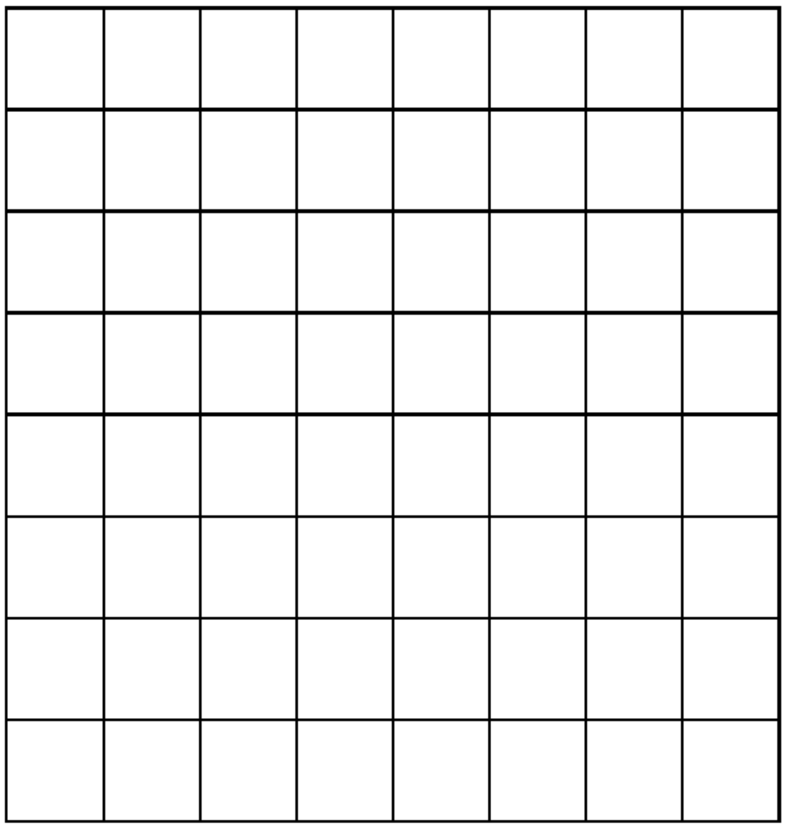
1 2 3 5 8 13

Number of tilings of a $2 \times n$ rectangle

The Fibonacci sequence!

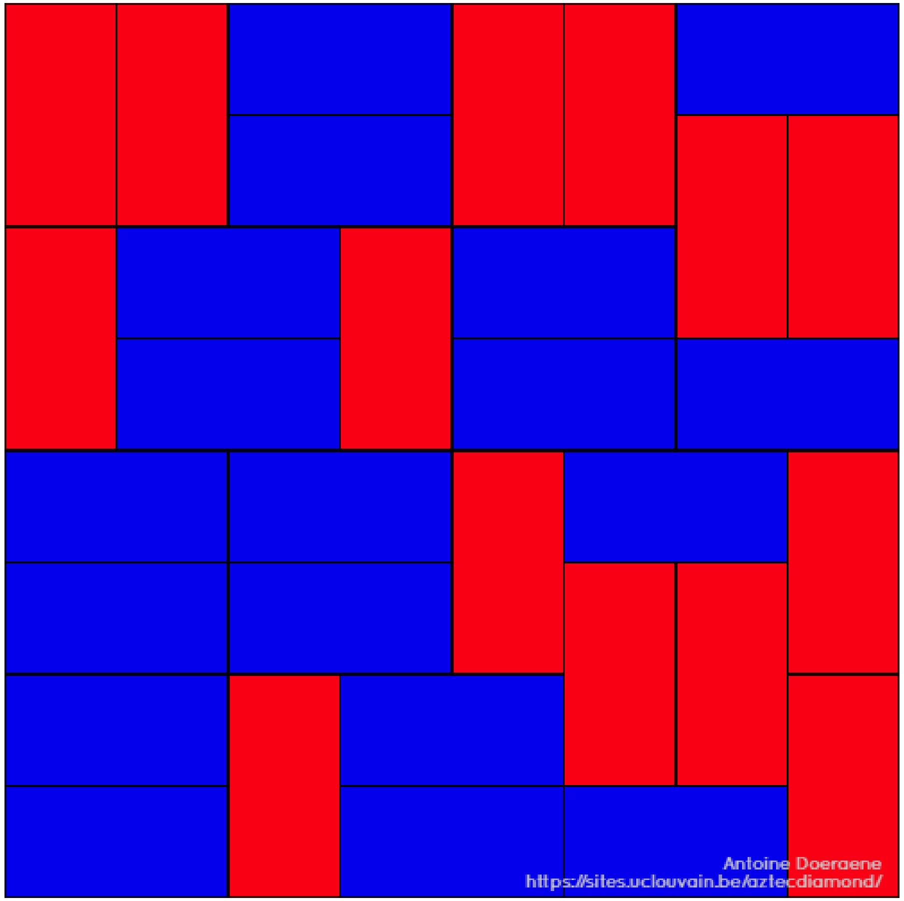
Checkerboard tilings

Second training example: tiling a square of size 8×8



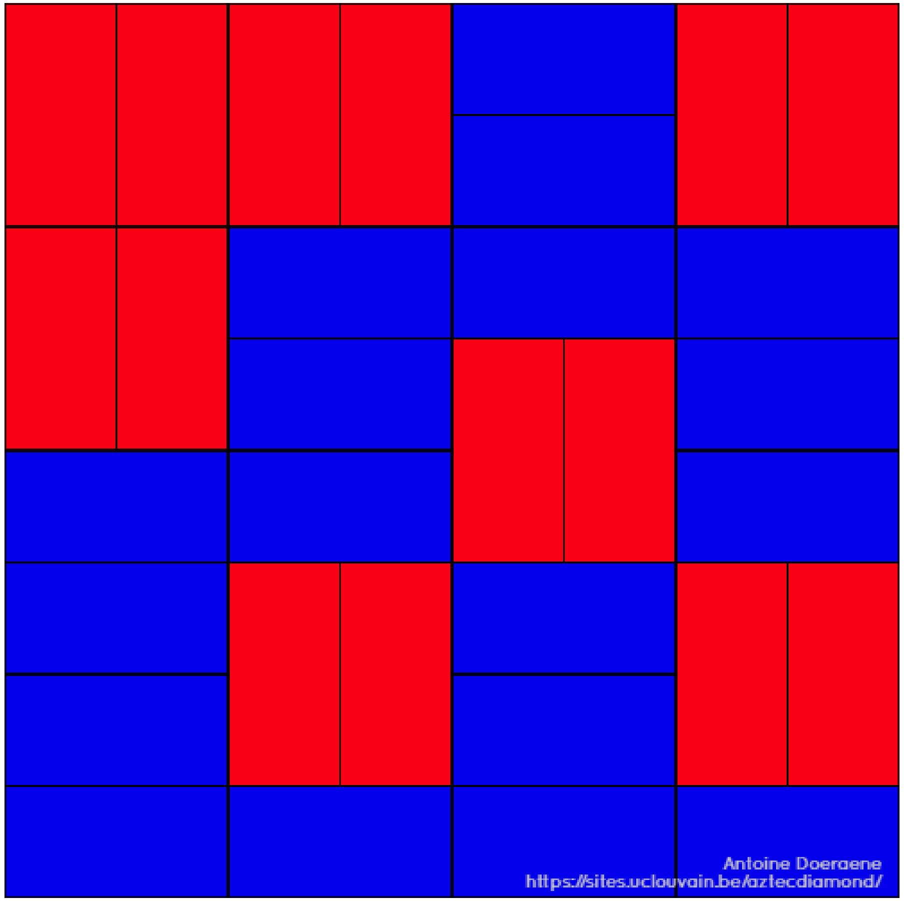
Checkerboard tilings

One tiling of the checkerboard



Checkerboard tilings

And another one ...

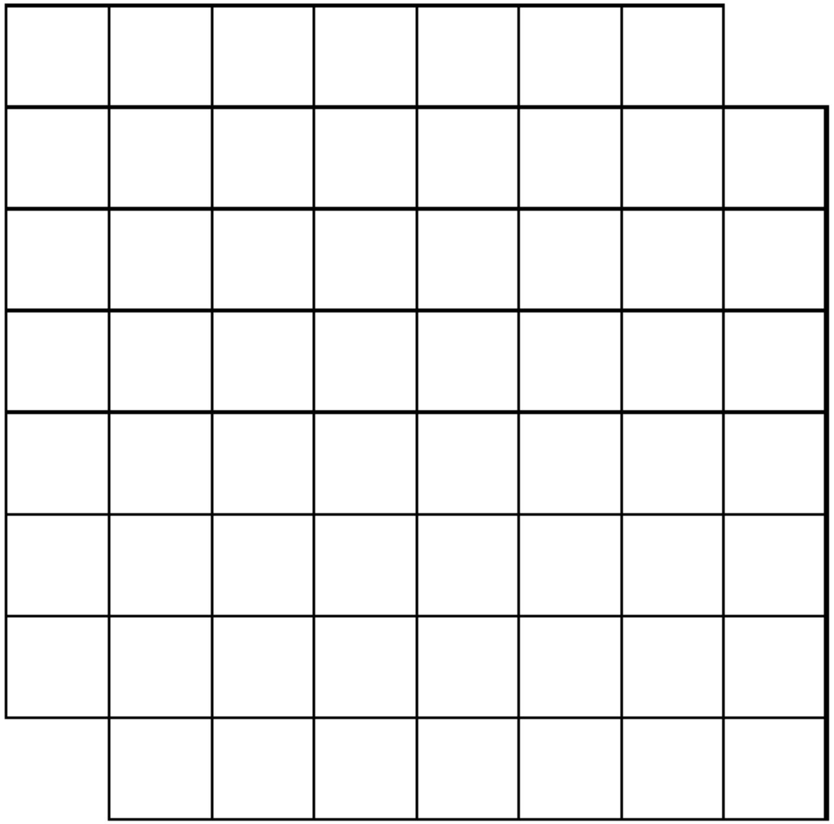


Number of tilings of a checkerboard?

✓ 12 988 816

Tilings of the mutilated checkerboard

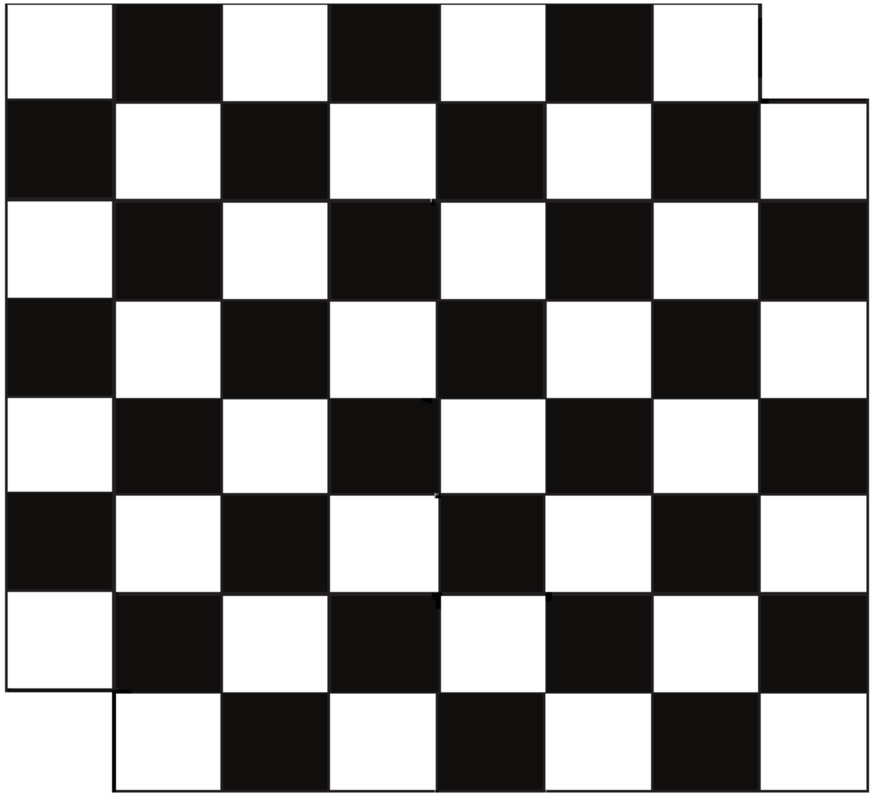
Third training example: two boxes removed

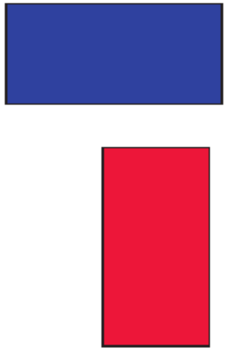


Number of tilings of the mutilated checkerboard?

Tilings of the mutilated checkerboard

Third training example: two boxes removed





Number of tilings of the mutilated checkerboard?

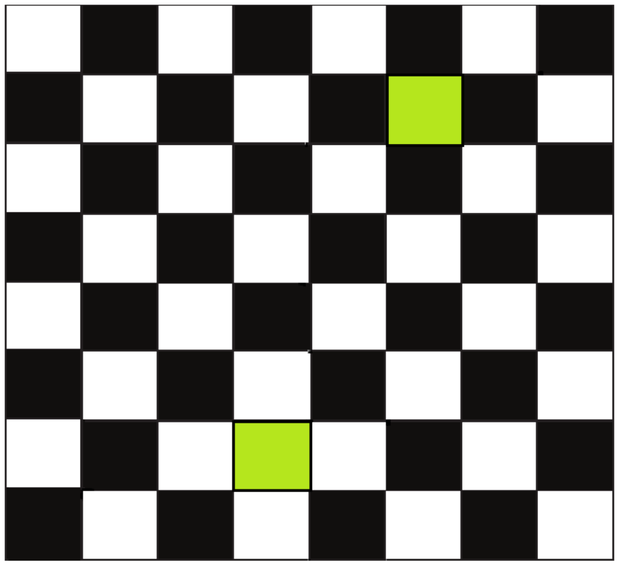
✓ **None !**

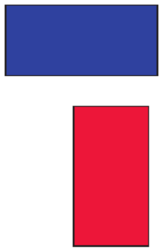
Tilings of the mutilated checkerboard

Third training example: two boxes removed

Theorem (Gomory 1973)

If we remove a white and a black box from the checkerboard, there exists always a tiling.





Tilings of the mutilated checkerboard

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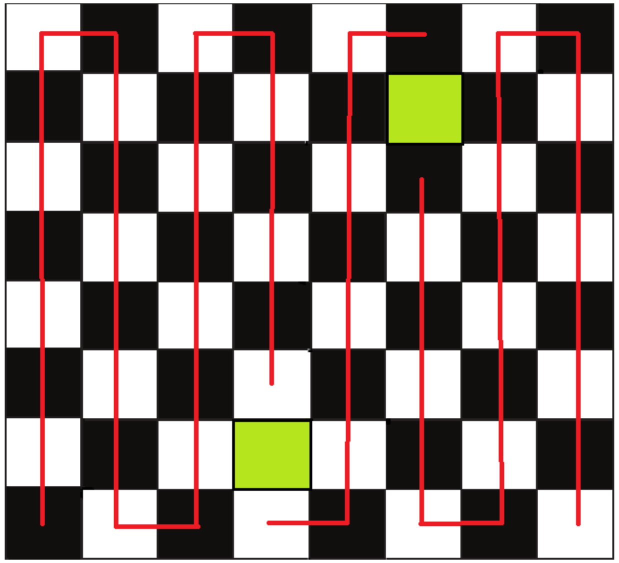
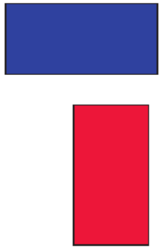




✓ **Take-home quiz:** what if we remove two white and two black boxes?

The Aztecs, their pyramids and their diamonds

An Aztec pyramid

The Aztecs, their pyramids and their diamonds

Mathematical version of an Aztec pyramid

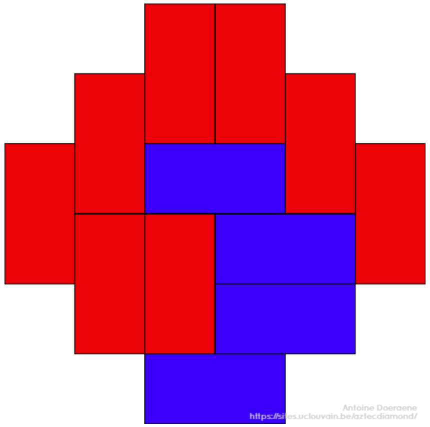
The Aztec diamond

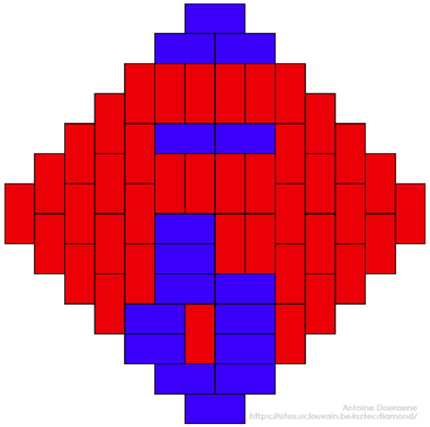
An Aztec diamond consists of two Aztec pyramids glued together

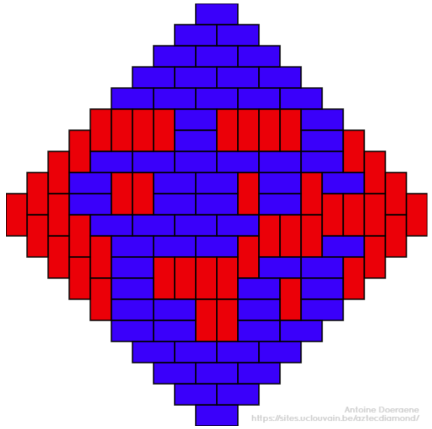
Tileable by domino's?

The Aztec diamond

An Aztec diamond consists of two Aztec pyramids glued together



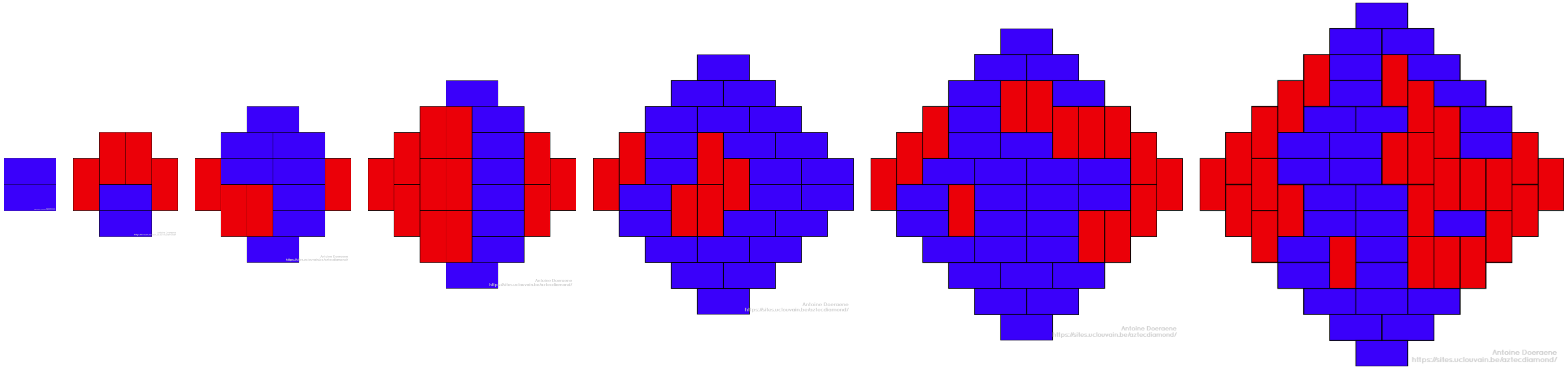




Tileable by domino's!

The Aztec diamond

Number of tilings



2 8 64 1024 32768 2 097 152 268 435 456

Theorem (Elkies-Kuperberg-Larsen-Propp 1992)

The number of tilings of the Aztec diamond of order n is $2^{\frac{n(n+1)}{2}}$

Number of tilings of the Aztec diamond

Some numbers

The number of tilings of the Aztec diamond of order 7 is

$$268\,435\,456$$

more than the number of possible Lotto combinations!

Number of tilings of the Aztec diamond

Some numbers

The number of tilings of the Aztec diamond of order 8 is

$$68\,719\,476\,736$$

ten times the number of humans on our planet!

Number of tilings of the Aztec diamond

Some numbers

The number of tilings of the Aztec diamond of order $\mathbf{24}$ is

$$\mathbf{203703597633448608626844568840937816105146839366593625063614044935438129976333670618339737}$$

more than the estimated number of particles in the universe!

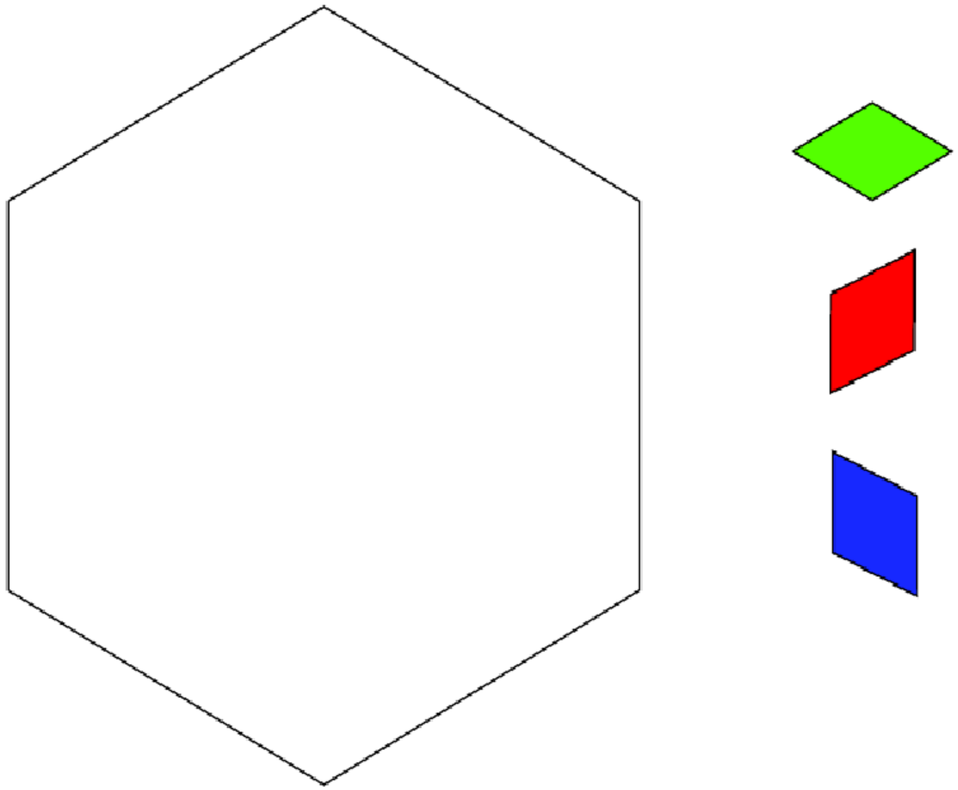
The reduced Aztec diamond

Number of tilings?

For each order, there is only one tiling, the horizontal one!

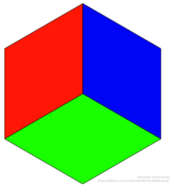
Tiling a hexagon with lozenges

Tiling a hexagon with lozenges of three types

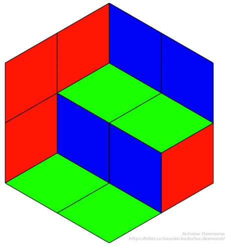


Tiling a hexagon with lozenges

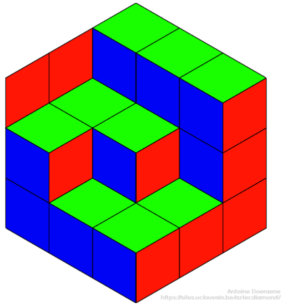
Number of tilings



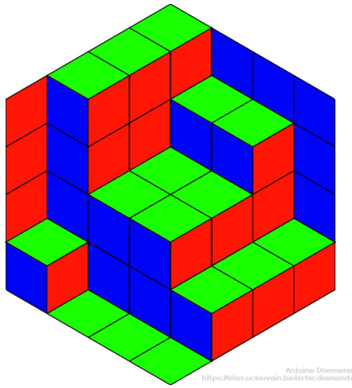
2



20



980



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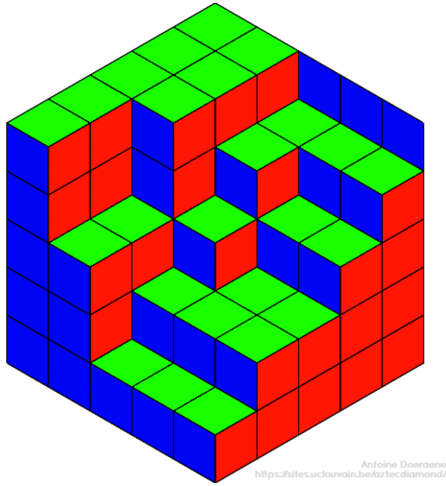

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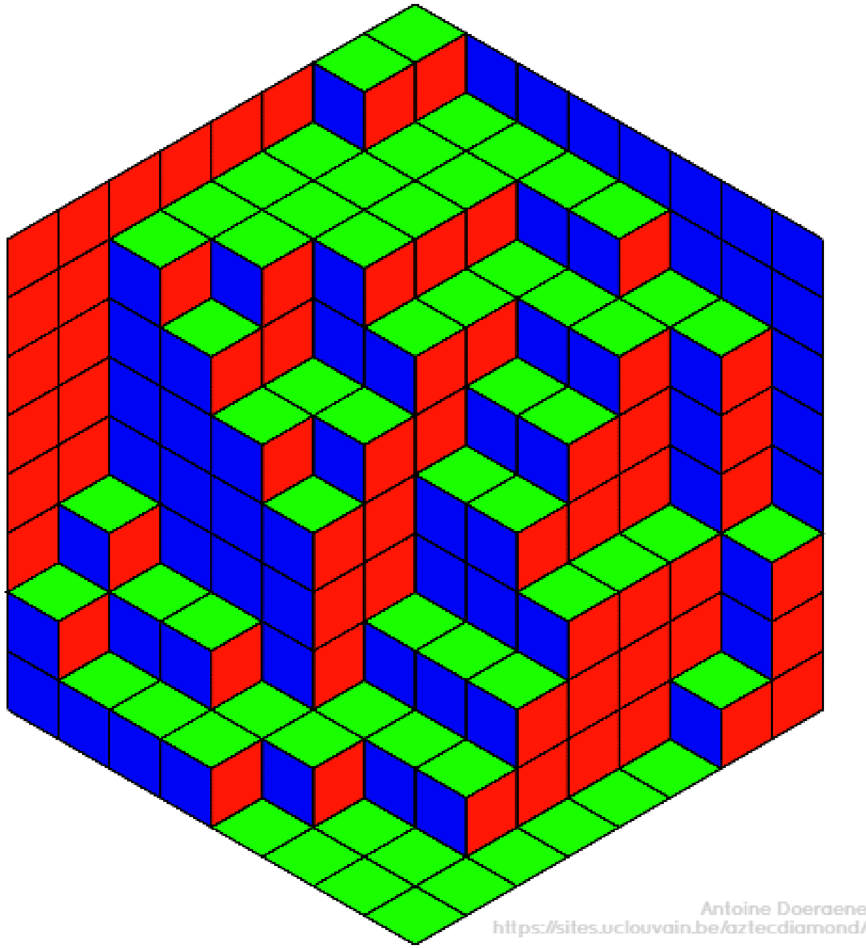
Theorem (Proctor 1988)

Explicit (but complicated) formula for the number of lozenge tilings of a hexagon.

Tiling a hexagon with lozenges

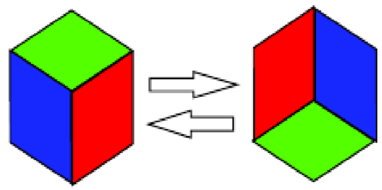
What do you see?



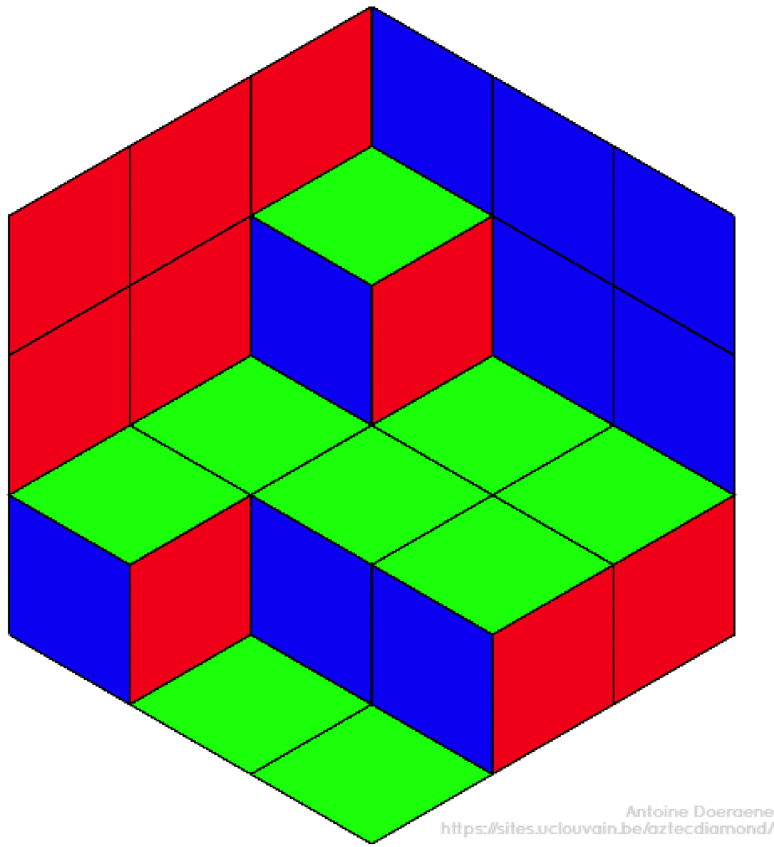
cubes in a corner (3d) or **lozenges in a hexagon (2d)**?

D'un pavage vers un autre

Un FLIP est une transformation de la forme



En 3d, un FLIP consiste à **enlever ou rajouter un cube**



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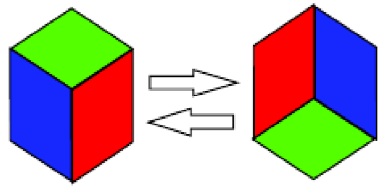

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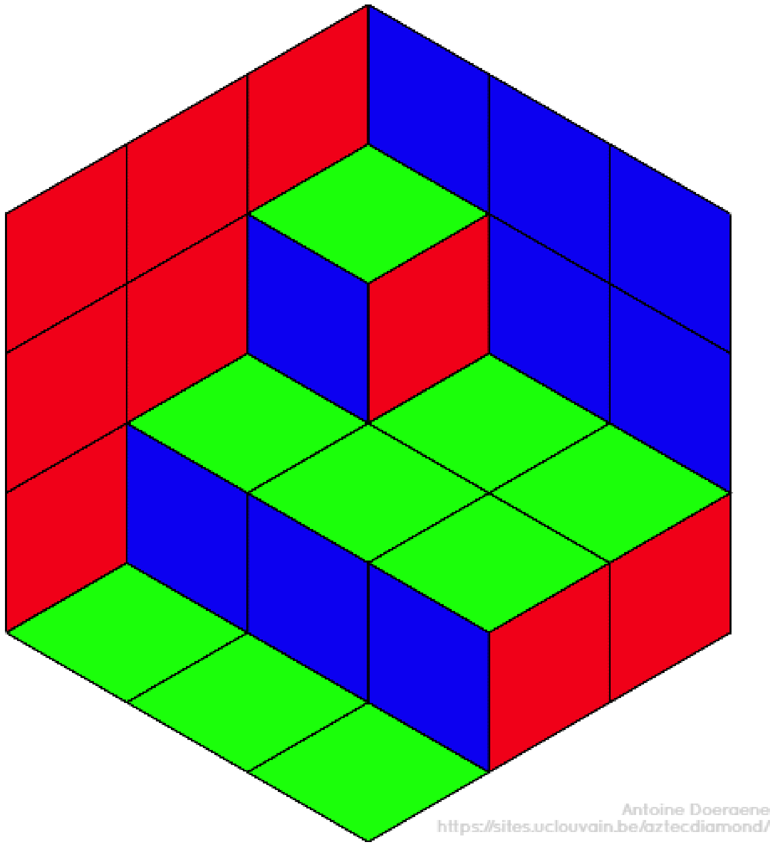


D'un pavage vers un autre

Theorem

Deux pavages par losanges d'un hexagone sont liés par une suite finie de flips.

Démonstration

Difficile en 2d, évident en 3d - on peut reformuler l'énoncé comme suit:

"Un rangement de cubes dans un coin peut toujours être obtenu à partir d'un autre en rajoutant ou en enlevant des cubes"

Random tiling of the Aztec diamond

Randomness ...

Let's equip the set of tilings with a **uniform probability measure**, i.e. each tiling is equally likely. Remarkably, despite the huge number of tilings, there is an **efficient algorithm to sample** a random tiling.

Uniform Diamond Generator

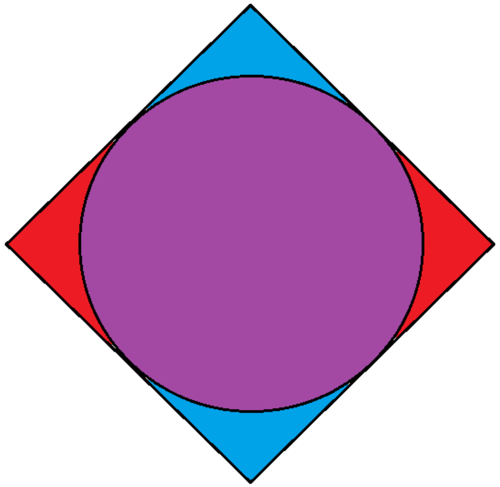
Diamond Order 100

Frozen region

We observe with high probability a FROZEN REGION close to the 4 corners and a FLUID REGION in the middle

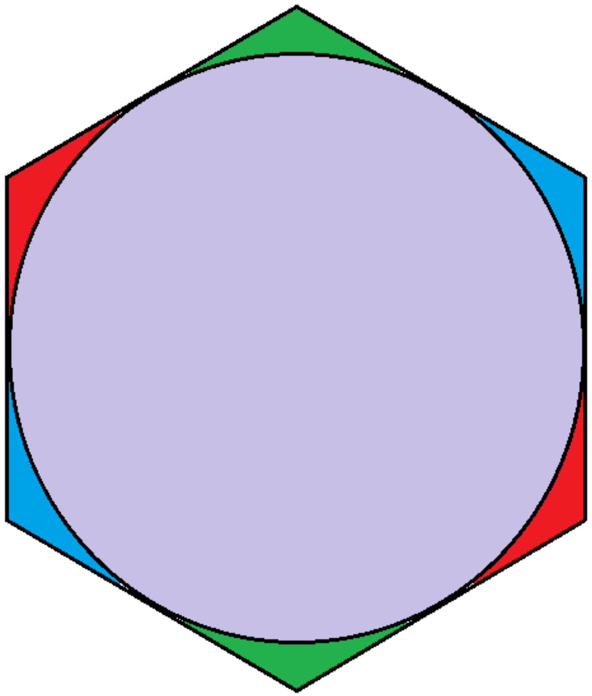
Arctic circle theorem (JOKUSCH, PROPP, SHOR 1998)

A uniform random tiling of a large Aztec diamond is very likely to have a frozen region and a fluid region, which are separated by a curve which is approximately the circle inscribed in the diamond.



Random lozenge tiling of a hexagon

Same phenomenon is observed for lozenge tilings of a hexagon

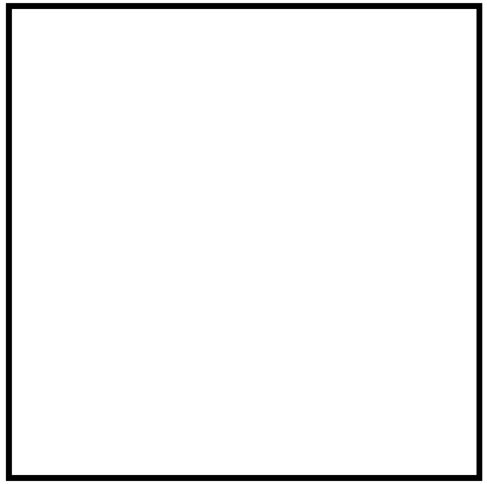


Hexagon Generator

First side size	15
Second side size	15
Third side size	15

Rectangle Generator

Width	40
Height	30



Aztec House Generator

Aztec n 30

Aztec h 30

Aztec Ring Generator

Inner order	20
Outer order	40

Questions

Some natural questions:

Macroscopic behavior

For which domains does one observe an arctic phenomenon, and what is the form of the arctic curve?

Microscopic behavior

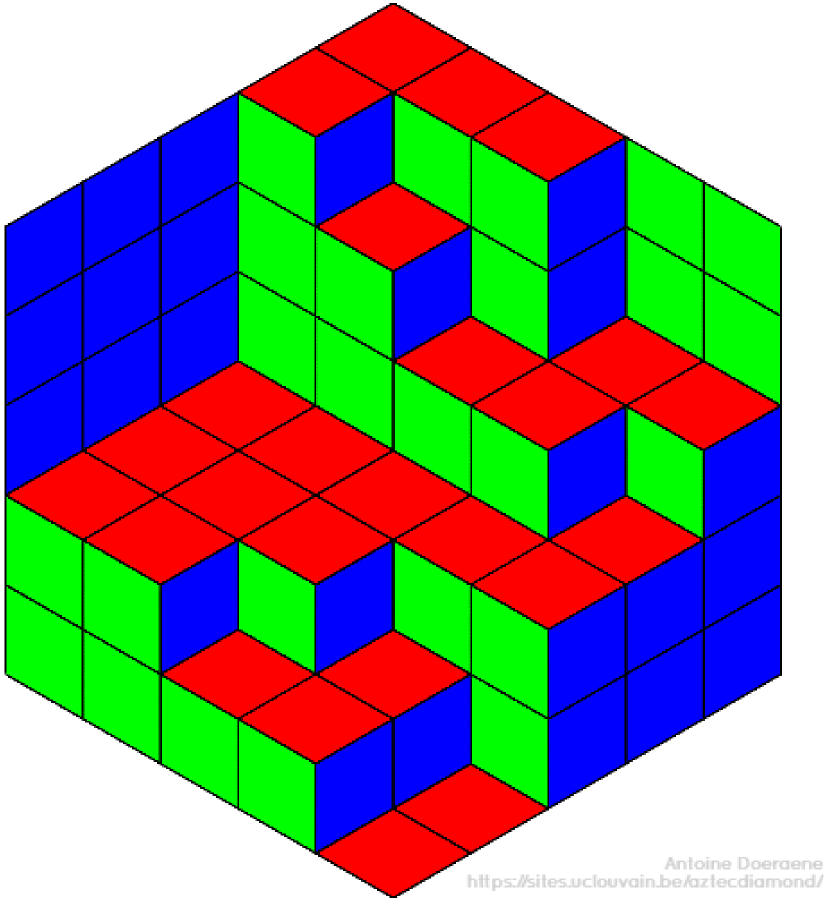
How can one describe fluctuations around the border?

Universality

To what extent are the phenomena similar for different domains?

Tilings and random walks

Let us manipulate the following hexagon tiling a bit ...



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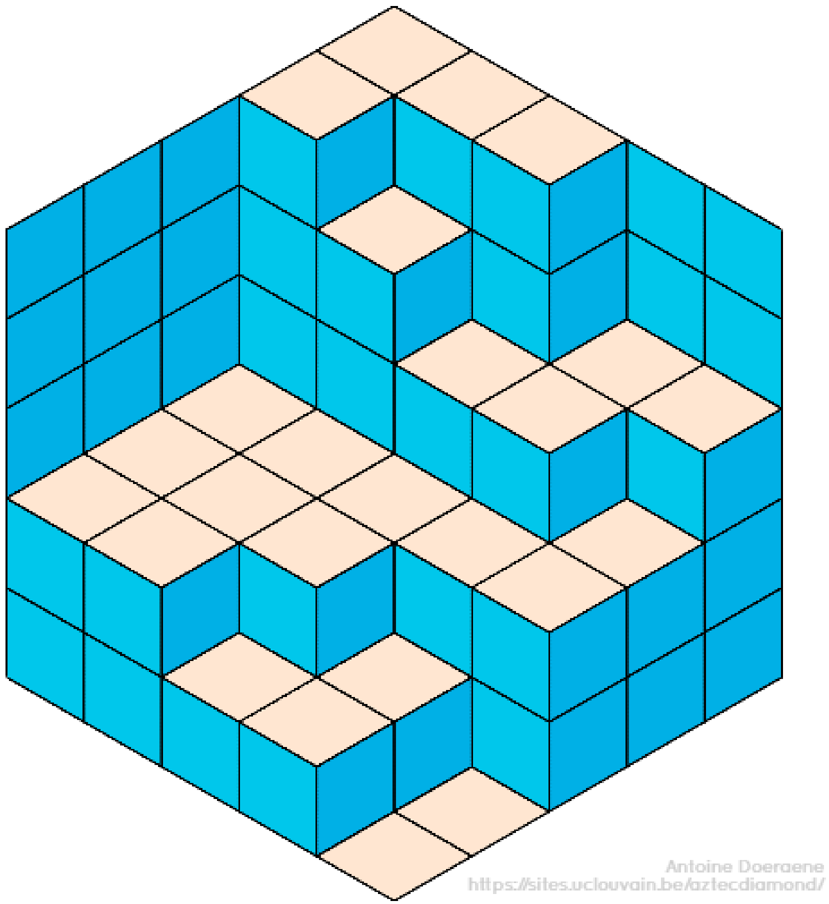

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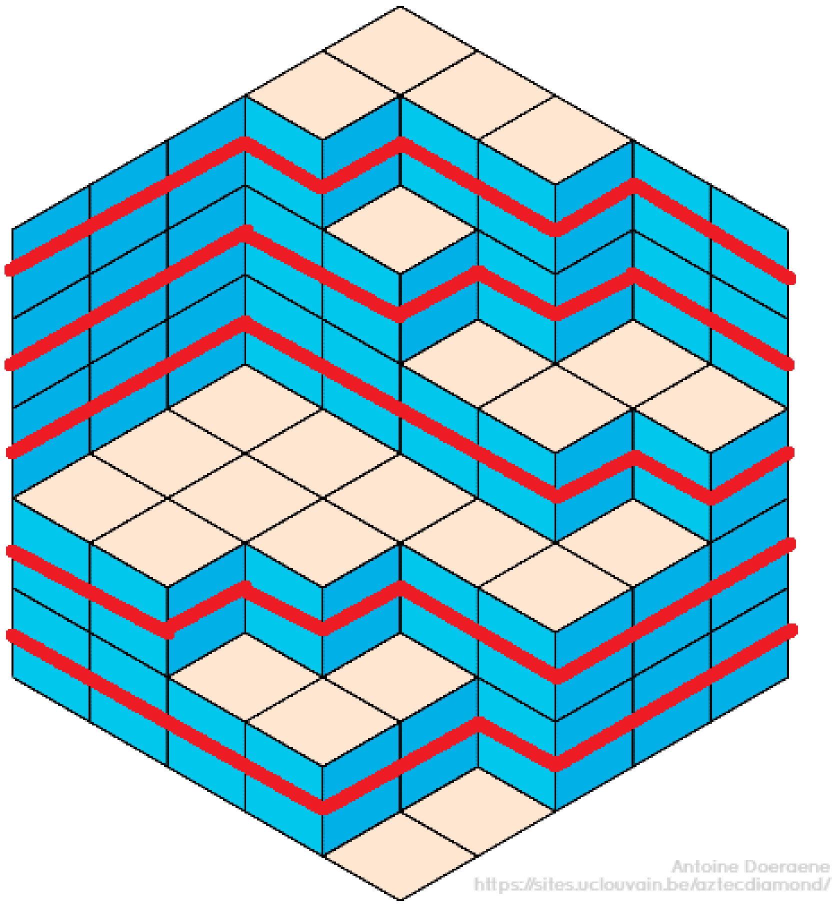

Tilings and random walks

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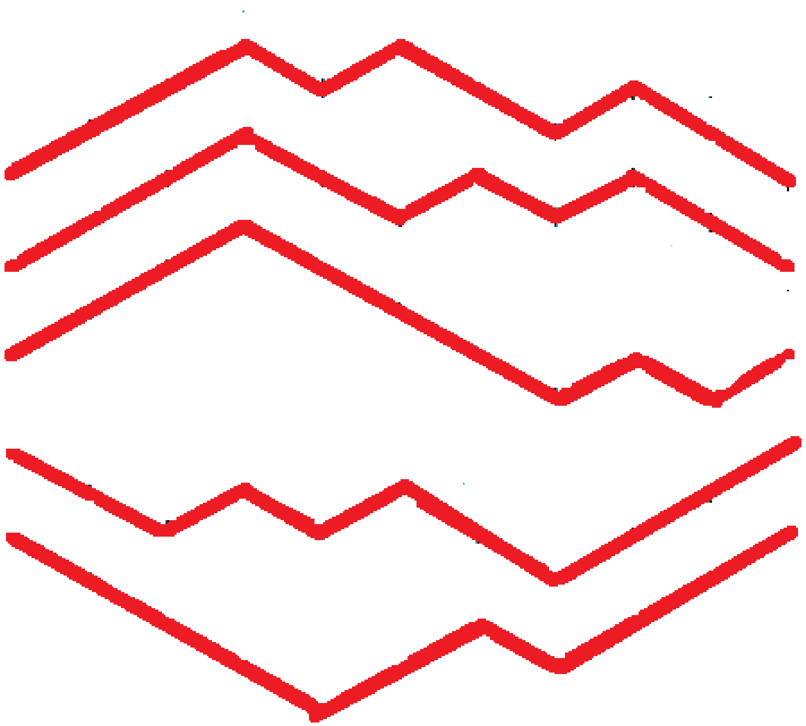


We see a configuration of up-down paths. At each step a path can go one unit up or down. For each such configuration of non-intersecting paths, we can recover a hexagon tiling.

Tilings and random walks

Bijection

There is a bijection between lozenge tilings of a hexagon and configurations of non-intersecting up-down paths.

Hexagon tiling vs random walks

A uniform random lozenge tilings of a hexagon can be interpreted as a uniform random configuration of **non-intersecting random walks**.

From a random walk to Brownian motion

1d Brownian motion

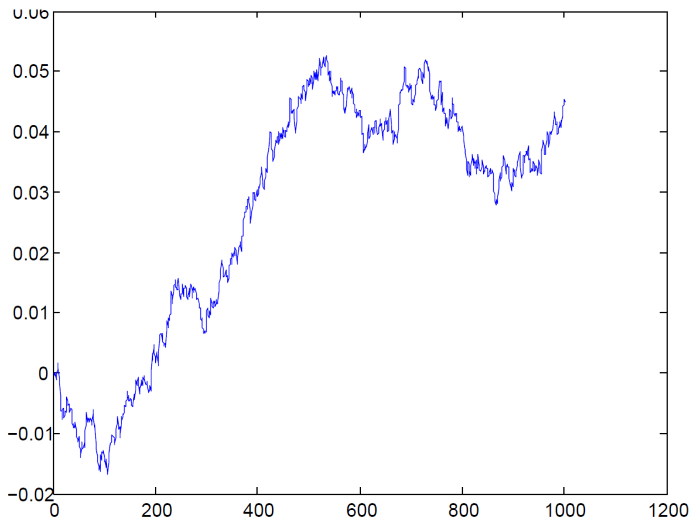
Random function $X : [0, T] \to \mathbb{R}$ characterized by the properties

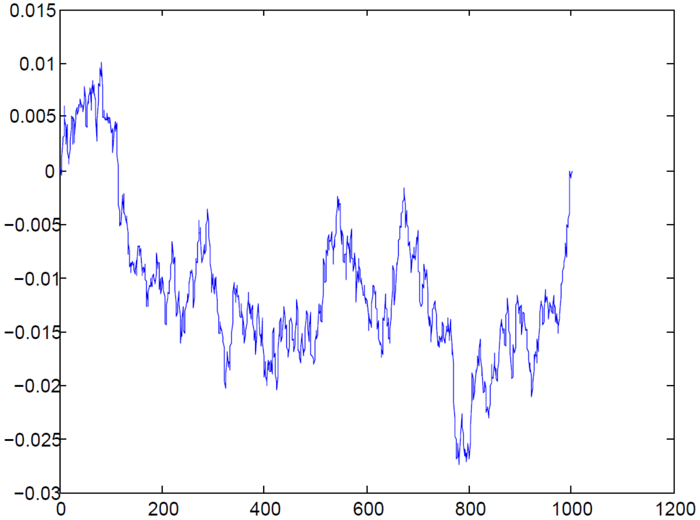
1. $X(0) = 0$
2. X is almost surely continuous
3. for all $0 < t_0 < t_1 < \ldots < t_m$, increments $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$, ..., $X(t_m) - X(t_{m-1})$ are independent
4. $X(t_1) - X(t_0) \sim \mathcal{N}(0, t_1 - t_0)$

Heuristics

Can be seen as a **continuous version of a random walk**
Has the **Markov property**: the future depends on the present but not on the past

Brownian motion and Brownian bridge





Brownian bridge

1d Brownian bridge

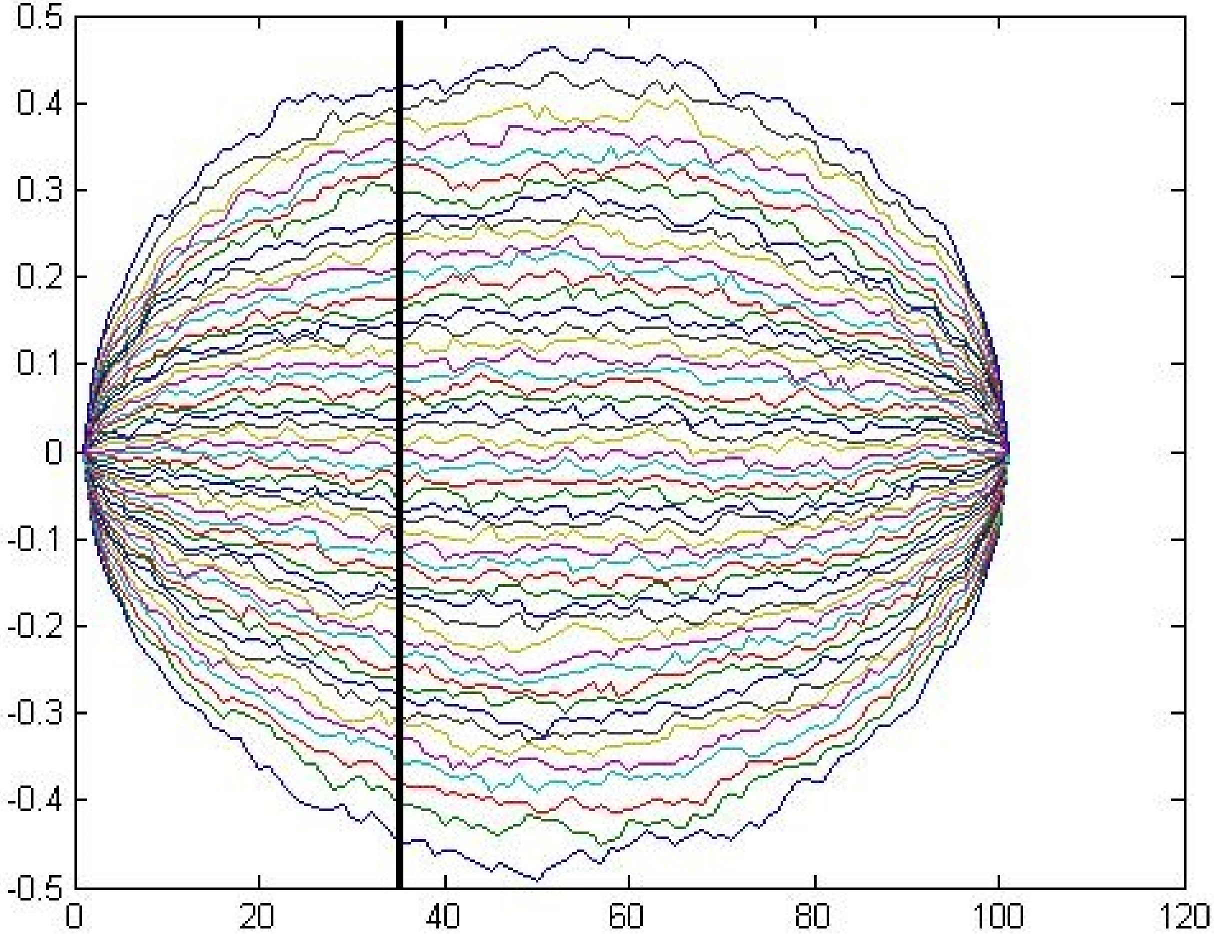
A **Brownian bridge** is a Brownian motion conditioned to start at $t = 0$ and end at $t = 1$ at the same point. It can be realized as

$$Y(t) = X(t) - tX(1)$$

Non-intersecting Brownian bridges

The natural continuous analogue of a hexagon tiling is now a collection of **non-intersecting Brownian bridges**.

Non-intersecting Brownian bridges



Joint probability distribution

A classical result of KARLIN-MCGREGOR (1959) gives, among others, the joint probability density function of the positions $x_1, \ldots, x_n$ of the bridges at a fixed time $t \in (0,1)$:

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{j=1}^{n} e^{-\frac{n}{2t(1-t)} x_j^2} dx_j$$

Qualitative interpretation

Two counteracting features:

- ✓ **Confining:** the positions $x_1, \ldots, x_n$ are unlikely to be large because of the Gaussian factors in the density
- ✓ **Repulsion:** the positions $x_1, \ldots, x_n$ repel each other because of the Vandermonde determinant

Random matrices

Random Wigner matrix

A random matrix is a **matrix filled with random variables.** If the matrix entries are independent and identically distributed (possibly up to some symmetry constraints), we speak of a WIGNER matrix.

Applications of random matrices

Various area's in physics, wireless communication, statistics, numerical analysis, number theory, modeling of social and political networks ...

Asymptotics of eigenvalues

A fundamental question in random matrix theory is to **understand the behavior of the eigenvalues** of a random matrix as the dimension n tends to infinity?

- ✓ Limiting density of eigenvalues?
- ✓ Correlations between eigenvalues
- ✓ Extreme eigenvalues?

The GUE

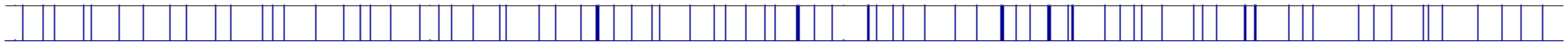
The **Gaussian Unitary Ensemble** consists of $n \times n$ Hermitian matrices with independent Gaussian entries: a GUE matrix H is of the form $H = M + M^*$ with

$$M_{i,j} = \mathcal{N}(0, \sigma^2) + i\mathcal{N}(0, \sigma^2)$$

For studying large n asymptotics, it is convenient to set $\sigma = \frac{1}{\sqrt{n}}$.

GUE eigenvalues

Eigenvalues of a 100×100 GUE matrix

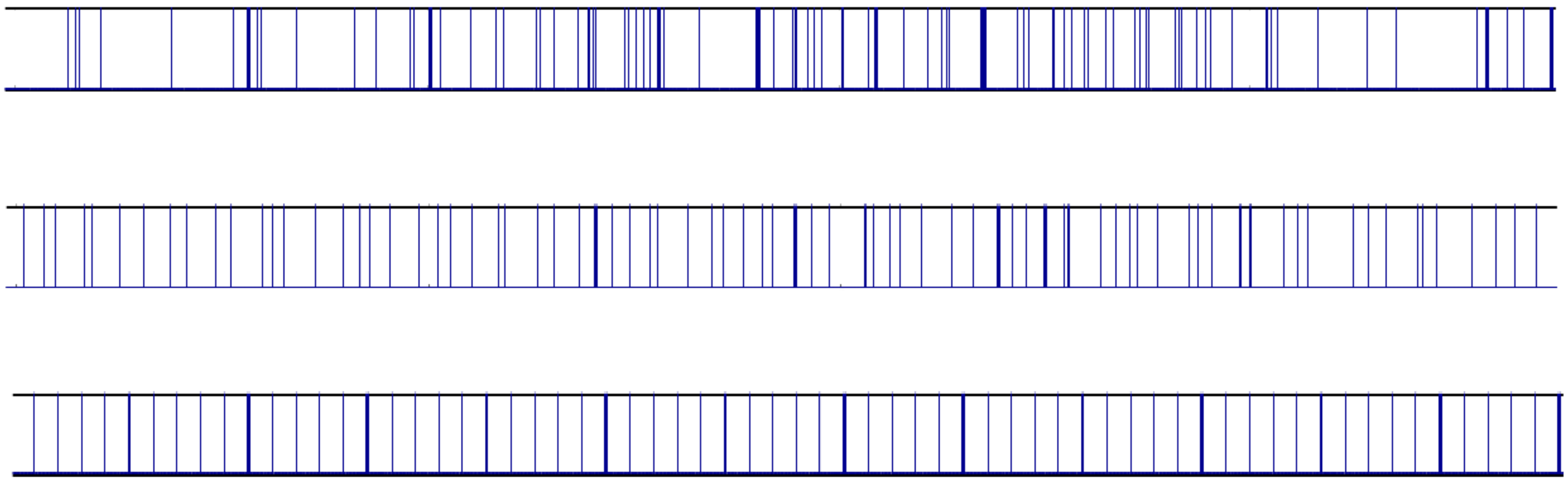


Distribution of GUE eigenvalues

$$\frac{1}{Z_n} \prod_{1 \le j < k \le n} (x_j - x_k)^2 \prod_{j=1}^{n} e^{-\frac{n}{2} x_j^2} dx_j$$

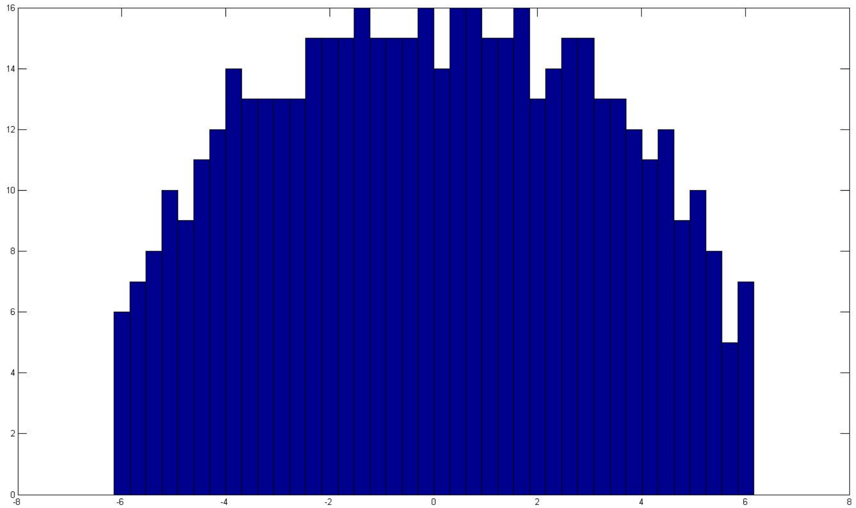
GUE eigenvalues

Comparison with independent random variables and equi-spaced points



GUE eigenvalues

Histogram of eigenvalues of a 500×500 GUE matrix



Wigner's semi-circle law

The **counting measure of the eigenvalues** converges almost surely to a probability measure with semi-circle density as $N \to \infty$.

Correlation functions

The m-point correlation functions can be expressed in a remarkably simple way:

$$R_m(x_1, \ldots, x_m) = \det \left(K_n(x_i, x_j)\right)_{i,j=1,\ldots,m},$$

for some function $K_n(x, y)$ which is called the **correlation kernel**

Determinantal point processes

A point process whose correlation functions have such a determinantal structure is called **determinantal point process**. Over the last decades, an impressive toolbox has been developed to study asymptotics for determinantal point processes.

Correlation kernel

The correlation kernel is given by

$$K_n(x,y) = e^{-\frac{n}{2}(x^2+y^2)} \sum_{j=0}^{n-1} p_j(x)p_j(y),$$

where p_j is the degree j **Hermite polynomial** characterized by the **orthogonality conditions**

$$\int_{\mathbb{R}} p_j(x)p_k(x)e^{-\frac{n}{2}x^2}dx = \delta_{jk}.$$

Christoffel-Darboux formula

The Christoffel-Darboux formula allows to evaluate the sum explicitly as

$$K_n(x,y) = \frac{\kappa_{n-1}}{\kappa_n} e^{-\frac{n}{2}(x^2+y^2)} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y},$$

where κ_j is the leading coefficient of p_j

GUE eigenvalues

Large n asymptotics

Understanding the **asymptotic behavior of the Hermite polynomials** allows us to understand the asymptotic behavior of the GUE eigenvalues.

- ✓ Scaling limits of the correlation kernel lead to **universal limiting kernels** like the sine and Airy kernel
- ✓ Gap probabilities can be expressed as **Hankel determinants**

Related models

Unitary Invariant Ensembles

Random matrix ensembles with eigenvalue distribution

$$\frac{1}{Z_n} \prod_{1 \le j < k \le n} (x_j - x_k)^2 \prod_{j=1}^{n} e^{-nV(x_j)} dx_j$$

- ✓ Are also determinantal point processes
- ✓ Correlation functions built out of orthogonal polynomials with respect to weight e^{-nV}

Related models

Beta Ensembles

Eigenvalue distribution

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{\beta} \prod_{j=1}^{n} e^{-nV(x_j)} dx_j$$

- ✓ $\beta = 1/T$ is a measure for repulsion
- ✓ Not determinantal in general, much harder to analyze, approach using stochastic operators

Questions

Problems in random matrix theory

Questions that have been investigated in the last 10 years or which are still to be investigated in various ensembles

- ✓ Extreme value distributions: what is the limit distribution of the largest eigenvalue of a random matrix?
- ✓ Gap probabilities: what is the probability that there are no eigenvalues in a given set?
- ✓ Rigidity: how far does an eigenvalue lie from its expected position?
- ✓ Behavior of eigenvectors?

References

Some links:

- ✓ Generating random tilings online - by Antoine Doeraene

  https://sites.uclouvain.be/aztecdiamond/

- ✓ Wikipedia

  http://en.wikipedia.org/wiki/Domino_tiling

- ✓ The mutilated checkerboard

  http://en.wikipedia.org/wiki/Mutilated_chessboard_problem

- ✓ http://www.claymath.org/library/senior_scholars/ stanley_ardila_tilings.pdf

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