University of Antwerp
Faculty of Science

The Homotopy Hypothesis

∞-groupoids vs topological spaces

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BSSM 2026

Groups

Definition

A group is a set G with:

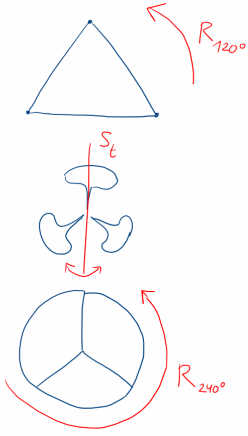
- a *multiplication* $* : G \times G \mapsto G : (g, h) \mapsto g * h$,
- a *neutral element* $e \in G$,
- for any $g \in G$, an *inverse* $g^{-1} \in G$.

and an element $e \in G$ such that

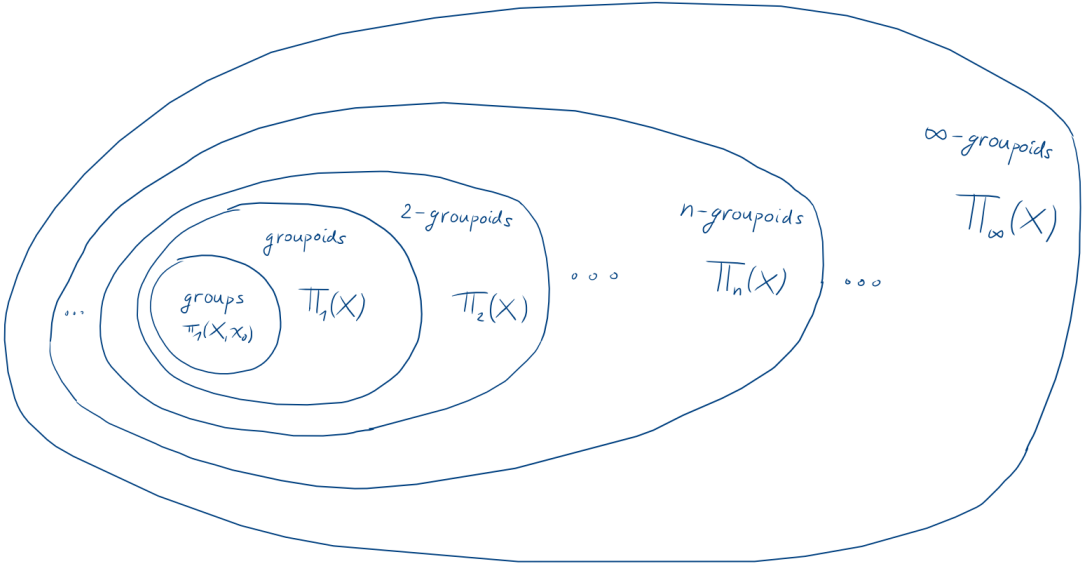
(1) $(g * h) * k = g * (k * h)$

(2) $g * e = e * g = g$

(3) $g * g^{-1} = g^{-1} * g = e$

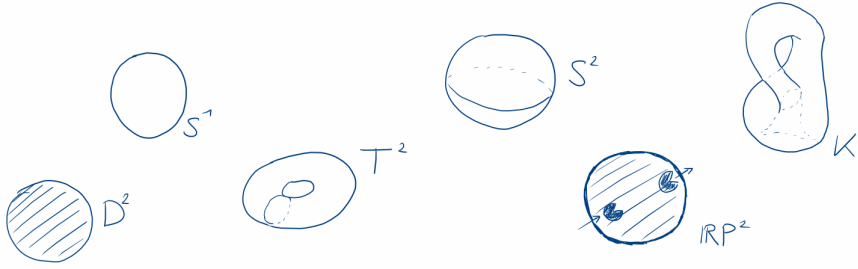


$$D_3 = \{\mathrm{id}, R_{120}, R_{240}, S_t, S_l, S_r\}$$


∞-groupoids
$\Pi_\infty(X)$
2-groupoids
groupoids
groups
$\pi_1(X, x_0)$
$\Pi_1(X)$
$\Pi_2(X)$
n-groupoids
$\Pi_n(X)$

Topological spaces

Roughly speaking, a (topological) space is just an abstract shape:



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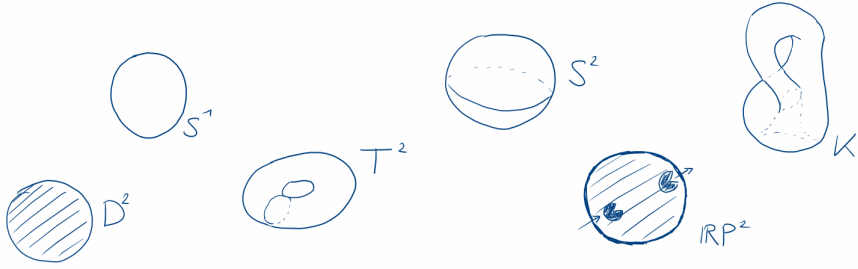

Algebraic Topology is the study of spaces up to *homotopy equivalence*.

"Definition"

Two spaces X and Y are homotopy equivalent if X can be deformed into Y by bending stretching and pinching, but without tearing, or creating or closing holes.
Notation: $X \simeq Y$.

Question: How can we classify spaces up to homotopy equivalence?

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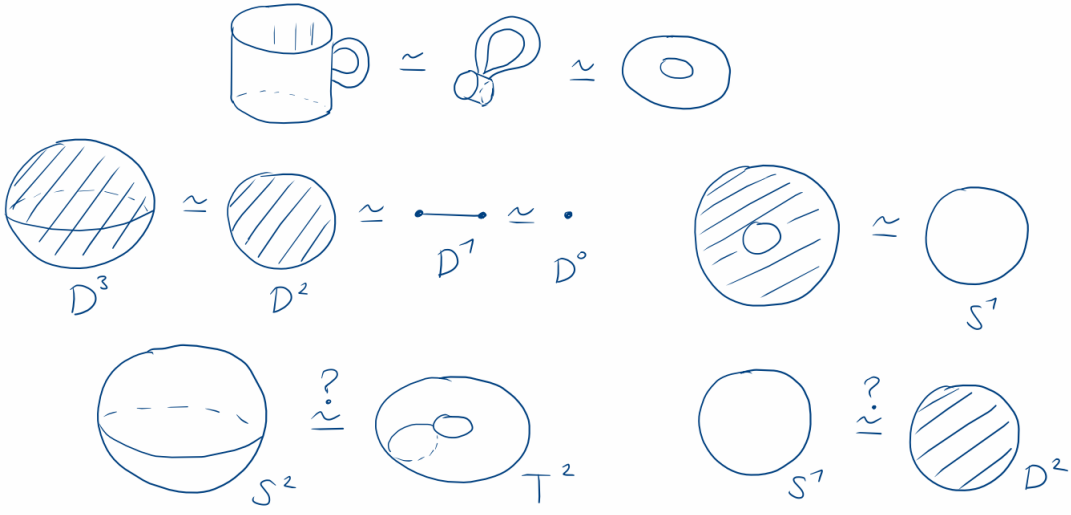

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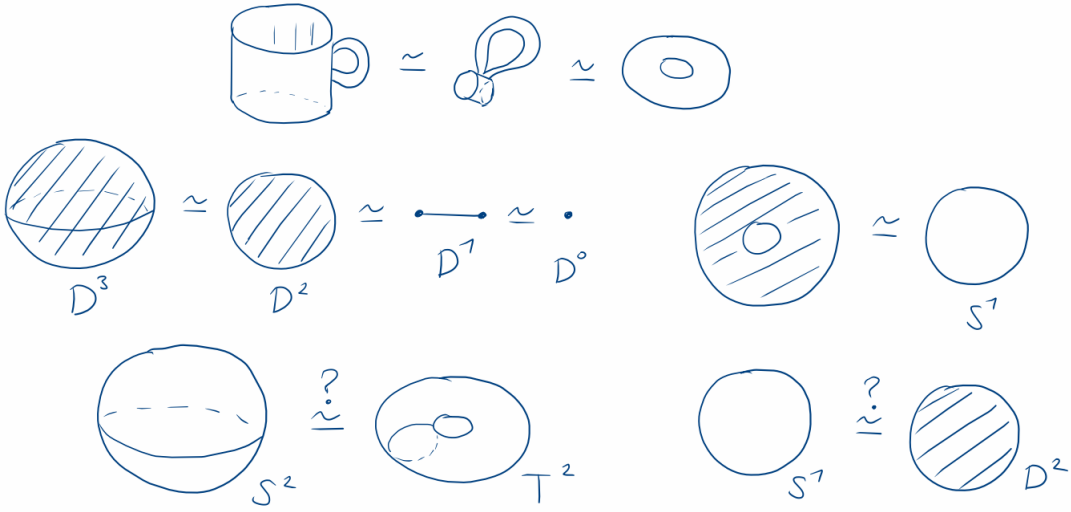

This is a very difficult problem in general! To help us, we consider an invariant $\pi_1(X)$:

$$X \simeq Y \quad \Rightarrow \quad \pi_1(X) \cong \pi_1(Y)$$

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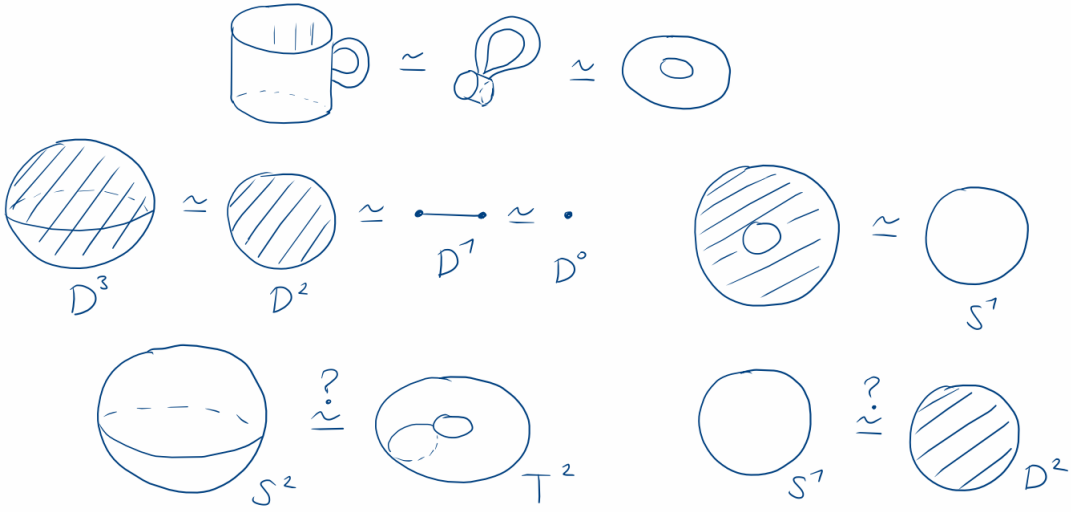

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$$X \not\simeq Y \quad \Leftarrow \quad \pi_1(X) \not\cong \pi_1(Y)$$

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Let X be a space, and $x, y \in X$ points on it.

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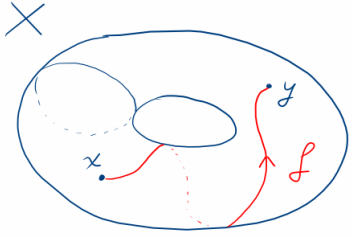
with $f(0) = x$ and $f(1) = y$.



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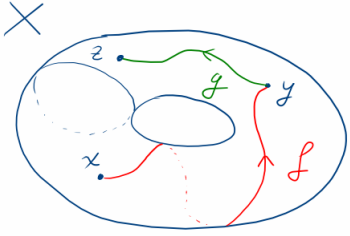
We define the composition of paths as:

$$f * g : [0,1] \to X : t \mapsto \begin{cases} f(2t) & \text{if } t \leq \frac{1}{2} \\ g(2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$



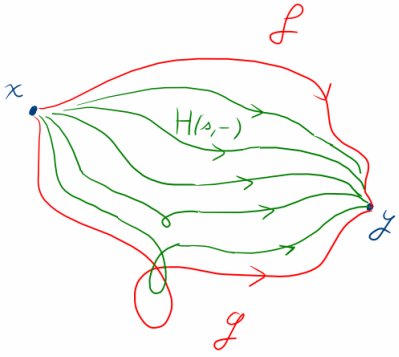
We consider paths up to continuous deformation:

Definition

Given paths f and g from x to y, a homotopy from f to g is a continuous map

$$H : [0,1] \times [0,1] \to X$$

such that $H(0,t) = f(t)$ and $H(1,t) = g(t)$, and $H(s,0) = x$ and $H(s,1) = y$.



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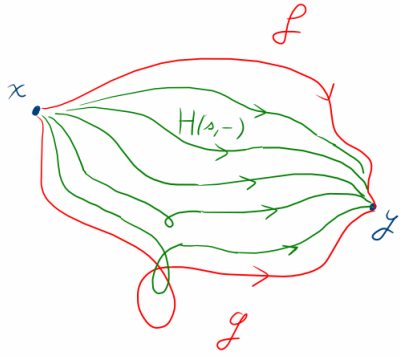

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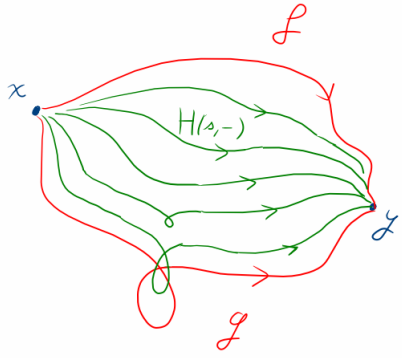
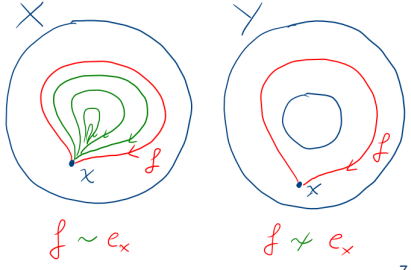


Homotopy allows us to detect holes in a space.



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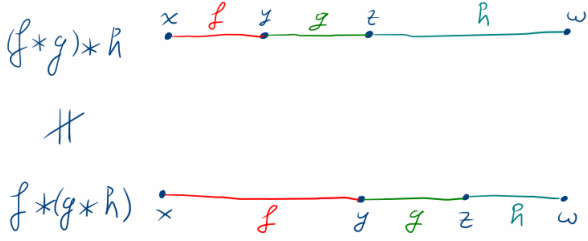

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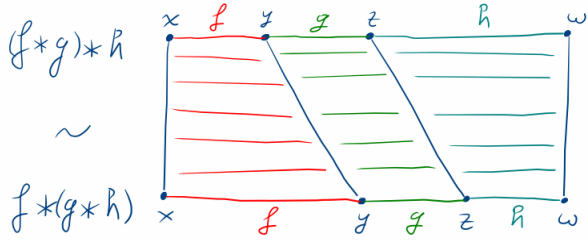

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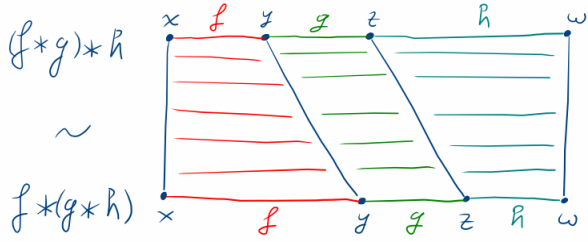

Further consider the constant path e and inverse path $\overline{f}$ of f:

$$e : [0,1] \to X : t \mapsto x_0 \qquad \text{and} \qquad \overline{f} : [0,1] \to X : t \mapsto f(1-t)$$

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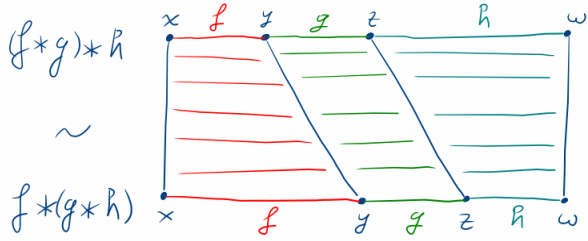

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Then:

(1) $(f * g) * h \sim f * (g * h)$

(2) $f * e \sim e * f \sim f$

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This sure looks a lot like a group!

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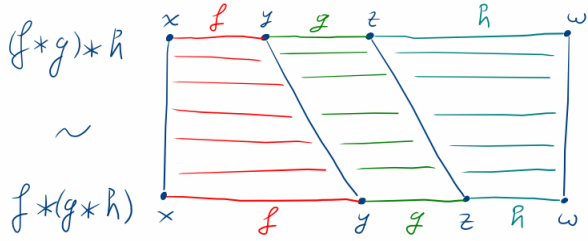

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Then:

(1) $([f] * [g]) * [h] = [f] * ([g] * [h])$

(2) $[f] * [e] = [e] * [f] = [f]$

(3) $[f] * [\overline{f}] = [\overline{f}] * [f] = [e]$

This sure *is* a group!

Definition

The fundamental group of X at a point x_0 is

$$\pi_1(X, x_0) = \{[f] \mid f \text{ path from } x_0 \text{ to } x_0\}$$

where $[f] = [g]$ whenever $f \sim g$.

Examples:

$\pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$

$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$

$\pi_1(K) \cong \langle a, b \mid ab = b^{-1}a \rangle$

$\pi_1(D^2) = 0$

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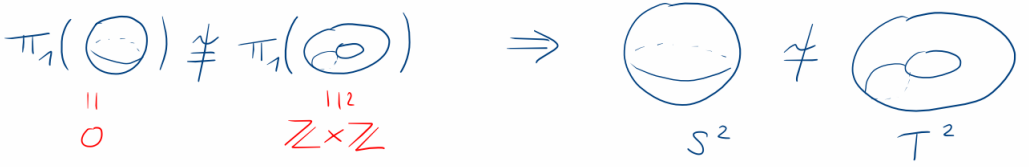
The fundamental group $\pi_1(X, x_0)$ is very useful to distinguish spaces:



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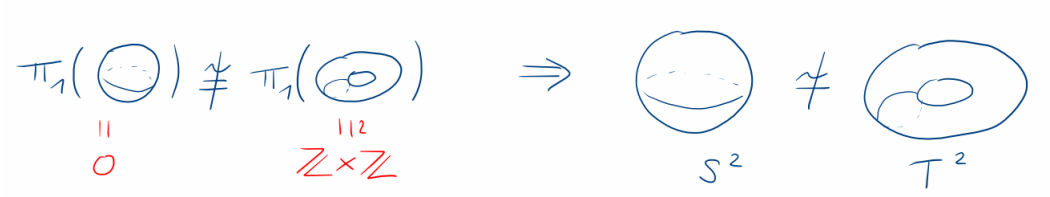

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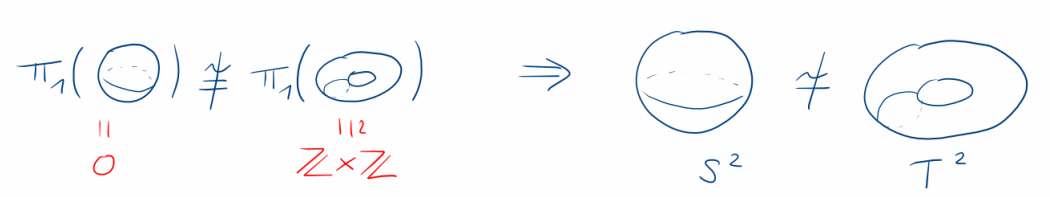

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Question: How can we upgrade $\pi_1(X, x_0)$ to a structure that captures more information?

Upgrading the fundamental group

Problem 1: $\pi_1(X, x_0)$ forgets all connected components of X that don't contain x_0.

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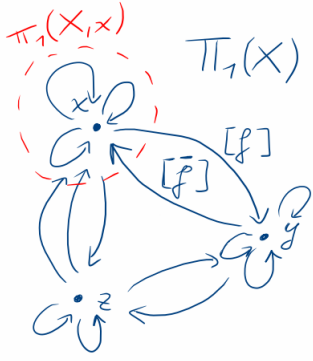
The fundamental groupoid of X is the directed graph $\Pi_1(X)$ with

- a vertex for every point x of X, and
- an arrow $x \to y$ for every **homotopy class** $[f]$ of continuous paths f from x to y.



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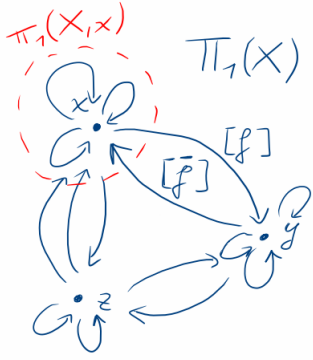

Definition

A groupoid is a directed graph with a composition of arrows

$$x \xrightarrow{f} y \xrightarrow{g} z \quad \mapsto \quad x \xrightarrow{f*g} y$$

that is associative, has a neutral element id_x for every vertex x, such that every arrow $f : x \to y$ has an inverse $\overline{f} : y \to x$.

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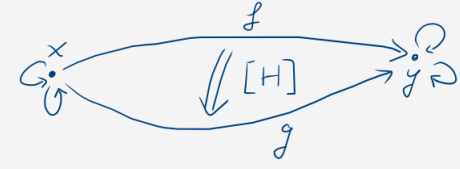

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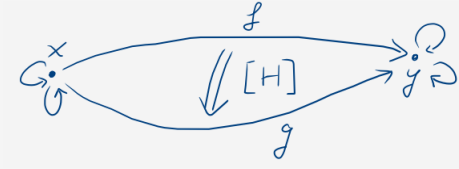

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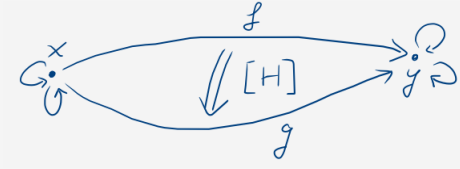

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Remark: Recall that the composition of paths is only associative up to homotopy!

More generally:

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- a composition of 1-arrows which is associative, has neutral elements and inverses, **only up to a 2-arrow**, e.g.

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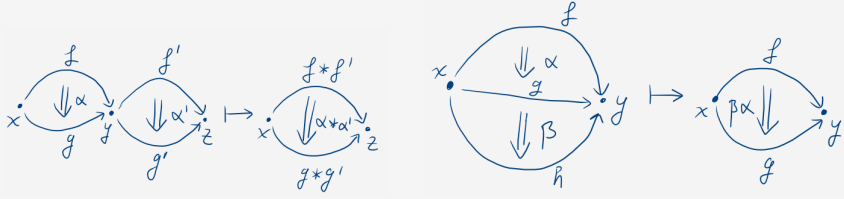
- *two* compositions of 2-arrows (horizontal and vertical), which are associative, have neutral elements and inverses + satisfy the "interchange law".



Problem 3: $\Pi_2(X)$ only captures the "2-dimensional data" of X.

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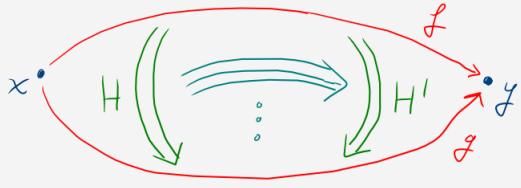
Definition

The fundamental ∞-groupoid of X is the ∞-dimensional directed graph $\Pi_\infty(X)$ with:

- a vertex for every point $x \in X$,
- an arrow $x \to y$ for every continuous path f from x to y,
- a 2-arrow $f \Rightarrow g$ for every homotopy of paths $H : f \sim g$,
- a 3-arrow $H \Rrightarrow H'$ for every homotopy of homotopies from $\alpha : H \sim H'$,
- ...



"Definition"

An ∞-groupoid is an ∞-dimensional directed graph, with for all $n \geq 1$:

- n different compositions of n-arrows,
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Warning

This is not a rigourous definition!
Instead, there exist many different equivalent "models" for this intuitive idea.
These often take a lot of work to define. Each has its own advantages and disadvantages.

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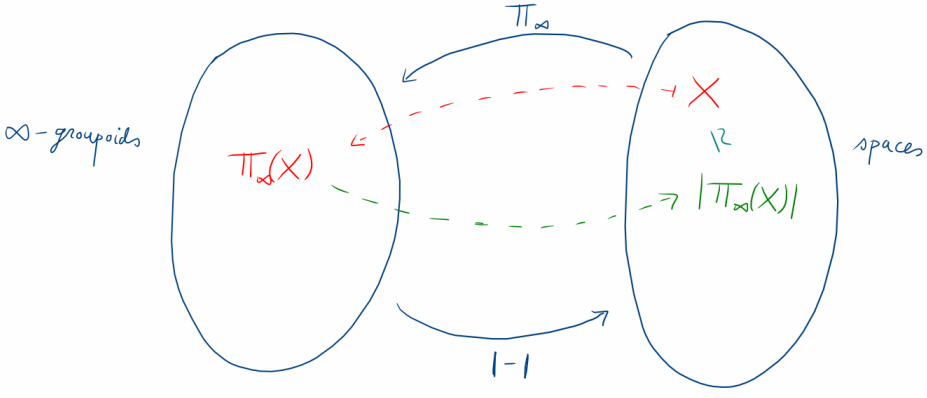

Depending on the chosen model for ∞-groupoids, this can be easy or hard to prove.

∞-groupoids	Proven?
Kan complexes	✓ [Joyal '02, Quillen '67]
Segal groupoids	✓ [Pellisier '03]
Grothendieck ∞-groupoids	✓ for $n \leq 3$ [Henry-Lanari '19]
strict ω-categories with weak inverses	✗

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Segal groupoids	✓ [Pellisier '03]
Grothendieck ∞-groupoids	✓ for $n \leq 3$ [Henry-Lanari '19]
strict ω-categories with weak inverses	✗

Any reasonable model for ∞-groupoids should satisfy the Homotopy hypothesis.

∞-categories and beyond

Analogy: Groups : Monoids $\rightsquigarrow$ ∞-groupoids : ∞-categories:

"Definition"

An ∞-category is an ∞-dimensional directed graph, with for all $n \geq 1$:

- n different compositions of n-arrows,
  which are associative, have neutral elements, and if $n \geq 2$, also inverses
  up to an $(n+1)$**-arrow** + various interchange laws.

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Analogy: Monoids : Algebras $\rightsquigarrow$ ∞-categories : linear ∞-categories

The comparison of various models for linear ∞-categories is an active field of research:

dg-categories, (stable) linear ∞-categories [Lurie], Complete dg-Segal spaces [Bermejo], Segal enriched categories [Bacard], Templicial modules [Lowen-Mertens].

Thanks for listening!