Rigidity theory for von Neumann algebras

Daniel Drimbe

KU Leuven

Brussels Summer School of Mathematics
Université Libre de Bruxelles
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Operator algebras

For a Hilbert space $\mathcal{H}$ and a linear map $T : \mathcal{H} \to \mathcal{H}$, we denote by $\|T\|$ its **norm**. Recall that $\|T\| = \sup\{\|T\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\}$.
The set $B(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}$ is a linear map $\mid \|T\| < \infty\}$ is called **the set of bounded operators on** $\mathcal{H}$.

Definition

We consider $*$-subalgebras $M \subset B(\mathcal{H})$, where the $*$-operation is the Hermitian adjoint.

- **C*-algebras:** norm closed $*$-subalgebras of $B(\mathcal{H})$.
  $\rightsquigarrow$ Unital commutative C*-algebras are of the form $C(X)$ where X is compact Hausdorff.
- **von Neumann algebras:** weakly closed $*$-subalgebras of $B(\mathcal{H})$.
  $\rightsquigarrow$ $T_i \to T$ weakly if and only if $\langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, for all $\xi, \eta \in \mathcal{H}$.
  $\rightsquigarrow$ Commutative von Neumann algebras are of the form $L^\infty(X, \mu)$ where (X, μ) is a measure space.

Close connections to group theory, representation theory, (continuous and measurable) dynamical systems, quantum information theory, etc.

More on von Neumann algebras

A $*$-subalgebra $M \subset B(\mathcal{H})$ is a **von Neumann algebra** if it is closed in the weak operator topology.

Examples

1. $B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.
2. $L^\infty(X) \subset B(L^2(X))$, where (X, μ) is a measure space.
3. The commutant $\mathcal{A}' := \{x \in B(\mathcal{H}) | \, xa = ax, \forall a \in \mathcal{A}\}$ of any set $\mathcal{A} \subset B(\mathcal{H})$ that is closed under adjoint.
4. **von Neumann's bicommutant theorem:**
   If $M \subset B(\mathcal{H})$ is a unital $*$-algebra, then M is a von Neumann algebra if and only if $M = (M')'$.

Discrete groups and operator algebras

Group C*-algebras and group von Neumann algebras

Let Γ be a countable (discrete) group.

- The left regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$: $(\lambda_g \xi)(h) = \xi(g^{-1}h)$.
- The **group algebra** $\mathbb{C}[\Gamma]$ is the linear span of $\{\lambda_g\}_{g\in\Gamma}$ and note that $\mathbb{C}[\Gamma] \subset B(\ell^2(\Gamma))$.
- Take the norm closure: (reduced) **group C*-algebra** $C_r^*(\Gamma)$.
- Take the weak closure: **group von Neumann algebra** $L(\Gamma)$.

We have $\mathbb{C}[\Gamma] \subset C_r^*(\Gamma) \subset L(\Gamma)$.

Remark. At each inclusion, information gets lost $\rightsquigarrow$ natural rigidity questions.

Open problems

- Free group factor problem: is $L(\mathbb{F}_n) \ncong L(\mathbb{F}_m)$ if $n \neq m$?
- Connes' rigidity conjecture: is $L(PSL_n(\mathbb{Z})) \ncong L(PSL_m(\mathbb{Z}))$ if $n \neq m$?

The structure and classification of operator algebras is highly non-trivial.

Dynamical systems and operator algebras

Measurable dynamics and von Neumann algebras

A measure preserving action $\Gamma \curvearrowright (X, \mu)$ gives rise to a von Neumann algebra $L^\infty(X) \rtimes \Gamma$.

- This von Neumann algebra contains $L^\infty(X)$ as a subalgebra.
- It contains Γ as unitary elements $\{u_g\}_{g \in \Gamma}$ and encode the group action: $u_g F u_g^* = g \cdot F$.

Orbit equivalence and W*-equivalence

Two measure preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are called:

- **conjugate:** $\exists$ a group isomorphism $\delta : \Gamma \to \Lambda$ and a measure preserving isomorphism $\Delta : X \to Y$ s.t. $\Delta(gx) = \delta(g)\Delta(x)$.
- **orbit equivalent:** $\exists$ a measure preserving isomorphism $\Delta : X \to Y$ s.t. $\Delta(\Gamma x) = \Lambda \Delta(x)$.
- **W*-equivalent:** $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$.

We have conjugacy $\rightsquigarrow$ orbit equivalence $\rightsquigarrow$ W*-equivalence.

Remark. At each step information gets lost $\rightsquigarrow$ natural rigidity questions.

Classification of von Neumann algebras

Problem

To what extent do $L(\Gamma)$ and $L^\infty(X) \rtimes \Gamma$ "remember" the underlying group Γ and group action $\Gamma \curvearrowright (X,\mu)$, respectively?

Remark. Γ infinite abelian $\implies L(\Gamma) \cong L^\infty(\hat{\Gamma}, \mathbf{Haar}) \cong L^\infty([0,1], \mathbf{Leb})$.
Terminology. A von Neumann algebra M is a **factor** if $M \ncong M_1 \oplus M_2$.

Proposition

- $L(\Gamma)$ is a factor if and only if Γ has infinite conjugacy classes (icc).
  Example. Wreath product groups $\Gamma = A \wr B := A^{(B)} \rtimes B$, free product groups $A * B$.
- $L^\infty(X) \rtimes \Gamma$ is a factor if $\Gamma \curvearrowright (X,\mu)$ is free and ergodic.
  Example. The **Bernoulli action** $\Gamma \curvearrowright (X_0,\mu_0)^\Gamma$ defined by $g \cdot (x_h)_{h\in\Gamma} = (x_{g^{-1}h})_{h\in\Gamma}$.

Murray and von Neumann, 1936-1943:

1. $\exists!$ approximately fin. dim. factor $R = \overline{\bigotimes_{n\in\mathbb{N}} \mathbb{M}_2(\mathbb{C})}^{\text{weakly}}$ (the **hyperfinite II_1 factor**).
2. $L(\mathbb{F}_2) \ncong R$, where $\mathbb{F}_2$ is the free group on two generators.

The amenable case

Definition

A group Γ is **amenable** if its left regular representation admits almost invariant vectors, i.e. there is a sequence of unit vectors $(\xi_n)_n \subset \ell^2(\Gamma)$ such that $\|\lambda_g(\xi_n) - \xi_n\|_2 \to 0$, for any $g \in \Gamma$.

Examples. Solvable (e.g. abelian) groups.

Remark. To see that $\mathbb{Z}$ is amenable, take $\xi_n = \frac{1}{\sqrt{n}}\mathbf{1}_{\{1,2,\ldots,n\}} \in \ell^2(\mathbb{Z})$ and note that $\|\lambda_m(\xi_n) - \xi_n\|_2^2 = \frac{2m}{n} \to 0$ as $n \to \infty$, for any $m \in \mathbb{Z}$.

Theorem (Connes, 1976)

- $L(\Gamma) \cong R$, for every icc amenable group Γ.
- $L^\infty(X) \rtimes \Gamma \cong R$, for every free ergodic action $\Gamma \curvearrowright (X, \mu)$ of an infinite amenable group.

Remark.

- Striking lack of rigidity: any info of Γ is lost when passing to von Neumann algebras.
- The classification of amenable C*-algebras is still very active (Stuart White, Wilhelm Winter, etc.).

The non-amenable case

Definition

A group Γ has **Kazhdan's property (T)** if any unitary representation of Γ with almost invariant vectors has non-zero invariant vectors.

Examples. Lattices in higher rank simple Lie groups, e.g. $SL_n(\mathbb{Z}), n \geq 3$.

Connes, 1980: If Γ is an icc property (T) groups, then any automorphism of $L(\Gamma)$ that is close to the identity is inner.

Connes' rigidity conjecture, 1980s

If Γ and Λ are icc property (T) groups with $L(\Gamma) \cong L(\Lambda)$, then $\Gamma \cong \Lambda$.

Cowling-Haagerup, 1989: If $\Gamma < Sp(m,1)$ and $\Lambda < Sp(n,1)$ are uniform lattices such that $L(\Gamma) \cong L(\Lambda)$, then $m = n$.

Popa's strong rigidity theorem, 2004: If $G_i = \mathbb{Z}/2\mathbb{Z} \wr \Gamma_i$ where Γ_i is an icc property (T) group for any $i \in \{1,2\}$ with $L(G_1) \cong L(G_2)$, then $G_1 \cong G_2$.

Popa's deformation/rigidity theory and W*-superrigidity

Definition

A group Γ is called **W*-superrigid** if whenever $L(\Gamma) \cong L(\Lambda)$ for some group Λ, then $\Gamma \cong \Lambda$.

Remark. Property (T) is a group von Neumann algebra invariant: if $L(\Gamma) \cong L(\Lambda)$ with Γ has property (T), then Λ has property (T) as well.

Connes' rigidity conjecture, 1980s: Any icc property (T) group Γ is W*-superrigid.

- Famous open problem (e.g. $\Gamma = PSL_3(\mathbb{Z})$).

Popa's deformation/rigidity theory (2001-)

General idea: Study von Neumann algebras M that have a

- **deformation property** (e.g. $\mathrm{Aut}(M)$ is large).
- **rigidity property** (e.g. $M = L(\Sigma \wr \Gamma)$, where Γ has property (T)).

Combine these properties to derive structural results for M.

- Led to spectacular progress in the theory of von Neumann algebras and orbit equivalence.
- In particular, it led to the first examples of W*-superrigid groups.

W*-superrigidity: examples

The **generalized wreath product group** $\Sigma \wr_I \Gamma$ is $\Sigma^{(I)} \rtimes_\sigma \Gamma$, where $\Gamma \curvearrowright I$ and $\sigma_g((x_i)_{i\in I}) = (x_{g^{-1}i})_{i\in I}$.

Examples of W*-superrigid groups

Ioana-Popa-Vaes, 2010: Certain generalized wreath product groups ($\Gamma = \mathbb{Z}/2\mathbb{Z} \wr_{K/B} K$).
Berbec-Vaes, 2012: Left-right wreath product groups $\mathbb{Z}/2\mathbb{Z} \wr_{\mathbb{F}_n} (\mathbb{F}_n \times \mathbb{F}_n)$.
Chifan-Ioana, 2017: Certain amalgamated free product groups ($\Gamma = \Gamma_1 *_\Sigma \Gamma_2$).
Chifan-Diaz-D, 2020: Iterations of certain amalgamated free product groups and HNN-extension groups (e.g. $\Gamma = \Gamma_1 *_\Sigma \Gamma_2 *_\Sigma \cdots *_\Sigma \Gamma_n$) are W*-superrigid.
Chifan-Ioana-Osin-Sun, 2021: The first examples of icc property (T) groups that are W*-superrigid.

W^*-superrigidity: functorial results

Question

Is the W^*-superrigidity property closed with respect to direct products?
$\rightsquigarrow$ Yes, if the groups are wreath product groups.

Theorem (D, 2020)

If Γ_1 and Γ_2 are W^*-superrigid wreath product groups, then $\Gamma_1 \times \Gamma_2$ is W^*-superrigid.

- Main ingredient (product rigidity result):
  Let Γ_1, Γ_2 be non-amenable wreath product groups and Λ any group for which $L(\Gamma_1 \times \Gamma_2) \cong L(\Lambda)$. Then $\Lambda = \Lambda_1 \times \Lambda_2$ such that $L(\Gamma_1) \cong L(\Lambda_1)$ and $L(\Gamma_2) \cong L(\Lambda_2)$.

Theorem (Chifan-Diaz-D, 2021)

Let Γ be an icc property (T) hyperbolic group. If A is any W^*-superrigid group, then the left-right wreath product group $A \wr_\Gamma (\Gamma \times \Gamma)$ is W^*-superrigid.

Superrigidity for graph product groups, I

Graph product groups (Green, 1990)

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a finite simple graph. To any family of groups $\{\Gamma_v\}_{v\in\mathscr{V}}$, one can naturally associate the so-called **graph product group** $\mathscr{G}\{\Gamma_v\}_{v\in\mathscr{V}}$.

- $\mathscr{G}\{\Gamma_v\}_{v\in\mathscr{V}} = \times_{v\in\mathscr{V}}\Gamma_v$ if $\mathscr{G}$ is complete.
- $\mathscr{G}\{\Gamma_v\}_{v\in\mathscr{V}} = *_{v\in\mathscr{V}}\Gamma_v$ if $\mathscr{G}$ has no edges.

Question

Does there exist a non-trivial graph product group that is W*-superrigid?

Superrigidity for graph product groups, II

Theorem (Chifan-Davis-D, 2023)

For certain graph product groups $\Gamma = \mathscr{G}\{\Gamma_v\}_{v\in\mathscr{V}}$, where $\mathscr{G}$ is a flower shaped graph, the following holds: if $L(\Gamma) \cong L(\Lambda)$, where Λ is any non-trivial graph product group whose vertex groups are infinite, then $\Gamma \cong \Lambda$.

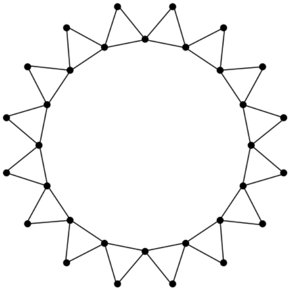

Figure: Flower shaped graph

Future questions

Problem

Identify new group constructions that are "recognizable" at the von Neumann algebra level.

Problem

1. Prove that $L(PSL_n(\mathbb{Z})) \ncong L(PSL_m(\mathbb{Z}))$, whenever $m \neq n$.
2. Show that $PSL_n(\mathbb{Z})$ with $n \geq 3$ is W*-superrigid.

W*-superrigidity for group actions

Consider the **Bernoulli action** $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ defined by $g \cdot (x_h)_{h\in\Gamma} = (x_{g^{-1}h})_{h\in\Gamma}$.

- If Γ is amenable, then $L^\infty(X_0)^\Gamma \rtimes \Gamma$ is isomorphic to the hyperfinite II_1 factor R.

Popa's strong rigidity theorem, 2004

Let Γ be a non-amenable icc group and $\Gamma \curvearrowright (X, \mu)$ a Bernoulli action. Let Λ be a property (T) group and $\Lambda \curvearrowright (Y, \nu)$ a free ergodic action.
If $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$, then $\Gamma \cong \Lambda$ and the actions are conjugate.

Definition

A group action $\Gamma \curvearrowright (X, \mu)$ is called **W*-superrigid** if whenever $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ for some free ergodic action $\Lambda \curvearrowright (Y, \nu)$, then $\Gamma \cong \Lambda$ and the actions are conjugate.

Theorem (Popa, 2003; Ioana, 2010; Ioana-Popa-Vaes, 2010)

If Γ is an icc non-amenable group such that Γ has property (T) or $\Gamma = \Gamma_1 \times \Gamma_2$, then any Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is W*-superrigid.

W*-superrigidity for actions on the hyperbolic plane

Consider the transitive infinite measure preserving action $PSL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2 = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ on the hyperbolic plane by fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Theorem (D-Vaes, 2021)

Let $\Gamma = PSL_2(\mathbb{Z}[S^{-1}])$, where S is a finite set of primes. The following hold:

1. If $S = \emptyset$, then $\Gamma \curvearrowright \mathbb{H}^2$ admits a fundamental domain.
2. If $|S| = 1$, then $L^\infty(\mathbb{H}^2) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ for uncountably many non-isomorphic Λ.
3. If $|S| \geq 2$, then $\Gamma \curvearrowright \mathbb{H}^2$ is W*-superrigid: if $L^\infty(\mathbb{H}^2) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$, we essentially have $\Gamma \cong \Lambda$ and the actions are conjugate.

- The first natural families of infinite measure preserving actions that are W*-superrigid.