Constant width bodies

Henri Anciaux

ULB

BSSM 2015 — August 6

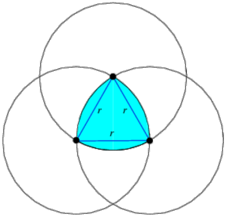
The width of a convex body

Let $\mathcal{C}$ be a convex body of $\mathbb{R}^n$ and $u \in \mathbb{S}^{n-1}$.

- The width $W(u)$ of $\mathcal{C}$ in the direction u is the distance between the two supporting hyperplanes of $\mathcal{C}$ normal to u
- When w does not depend on the direction u, $\mathcal{C}$ is said to have constant width.
- A simple example is the round ball B_W of radius $W/2$; but there are many others!

The Reuleaux triangle

It is obtained by intersecting three discs of equal radius centered at the vertices of an equilateral triangle.



The Reuleaux triangle is ubiquitous...

Onze-Lieve-Vrouwekerk, Bruges, Belgium

The Reuleaux triangle is ubiquitous...

Köln Triangle, Cologne, Germany

The Reuleaux triangle is ubiquitous...

A water valve cover, San Francisco, USA

Constant width bodies are everywhere...

Constant width bodies are everywhere...

The double normal property

Let $p \in \partial\mathcal{C}$ and $u \in N_p^i \partial\mathcal{C}$ (the inward normal cone of $\partial\mathcal{C}$ at p).
$\exists W(p) \in \mathbb{R}$ such that

$$q(p) := p + W(p)u \in \partial\mathcal{C}.$$

Proposition

$$\mathcal{C} \text{ has constant width } W$$

$$\Longleftrightarrow$$

$$-u \text{ is inward normal to } \partial\mathcal{C} \text{ at } q(p), \forall p \in \partial\mathcal{C}, u \in N_p^i \partial\mathcal{C}.$$

$$\text{i.e. } -u \in N_{q(p)}^i \partial\mathcal{C},$$

When it is the case, $W(p)$ is constant and equal to W.

CW $\Leftrightarrow$ double normal property

Assume $\gamma(s)$ is smooth, parametrized by arclength s. Denote by $(\vec{t}(s), \vec{n}(s))$ its Frénet frame, i.e. $\gamma'(s) = \vec{t}(s)$ and

$$\begin{cases} \vec{t}' & = & k\vec{n} \\ \vec{n}' & = & -k\vec{t}. \end{cases}$$

The function k is the curvature of γ.

$$\gamma \text{ is convex } \iff k \geq 0.$$

Moreover,

$$\gamma \text{ is strictly convex } \iff k > 0.$$

c. w. $\Rightarrow$ double normal property

If γ has c. w. W, there exists $a(s)$ and $\sigma : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\gamma(s) + W\vec{n}(s) + a(s)\vec{t}(s) = \gamma\big(\sigma(s)\big).$$

Differentiating,

$$\Big(1 - Wk(s) + a'(s)\Big)\vec{t}(s) + a(s)k(s)\vec{n} = \frac{d\sigma}{ds}\vec{t}(\sigma(s)).$$

c. w. $\Rightarrow$ $\vec{t}(s) \,||\, \vec{t}\big(\sigma(s)\big)$ $\Rightarrow$ a vanishes $\Rightarrow$ double normal property.

Double normal property $\Rightarrow$ c. w.

If γ has the double normal property, there exists $W(s)$ and $\sigma : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\gamma(s) + W(s)\vec{n}(s) = \gamma\big(\sigma(s)\big).$$

Differentiating,

$$\Big(1 - W(s)k(s)\Big)\vec{t}(s) + W'(s)k(s)\vec{n} = \frac{d\sigma}{ds}\vec{t}\big(\sigma(s)\big).$$

Double normal property $\Rightarrow \vec{t}(s) \,||\, \vec{t}\big(\sigma(s)\big) \Rightarrow W$ constant $\Rightarrow$ c. w.

$\longrightarrow$ This allows to extend the concept of width to convex bodies in complete Riemannian manifolds:

Definition

A convex $\mathcal{C}$ of a Riemannian manifold $(\mathcal{M}, g)$ will be said to have constant width if its enjoys the double normal property.

Yet another equivalent property

Proposition

$$\sup_{q\in\partial\mathcal{C}} \mathit{dist}(p,q) \text{ does not depend on } p$$

$$\iff$$

$$\mathcal{C} \text{ has constant width.}$$

When it is the case, the width is $\sup_{q\in\partial\mathcal{C}} \mathit{dist}(p,q)$.

Barbier's formula

Theorem

Let $\mathcal{C}$ a convex body of the plane with constant width W. Then

$$\mathcal{L}(\partial\mathcal{C}) = \pi W.$$

In the sphere

Theorem

Let $\mathcal{C}$ a convex body of $\mathbb{S}^2$ with constant width W. Then

$$\mathcal{L}(\partial\mathcal{C}) = \Big((2\pi - \mathcal{A}(\mathcal{C})\Big)\tan(W/2).$$

In hyperbolic plane

Theorem

Let $\mathcal{C}$ a convex body of $\mathbb{H}^2$ with constant width W. Then

$$\mathcal{L}(\partial\mathcal{C}) = \left(2\pi + \mathcal{A}(\mathcal{C})\right)\tanh(W/2).$$

Summarizing

Theorem

Let $c \in \{-1, 0, 1\}$ *and* $\mathbb{Q}^2_c$ *the "space form of curvature c". Let* $\mathcal{C}$ *be a convex body of* $\mathbb{Q}^2_c$ *with constant width* W*. Then*

$$\mathcal{L}(\partial\mathcal{C}) = \Big(2\pi + c\mathcal{A}(\mathcal{C})\Big)\mathcal{T}_c(W/2),$$

where

$$\mathcal{T}_c(t) = \begin{cases} \tan(t) & \textit{if } c = 1; \\ t & \textit{if } c = 0; \\ \tanh(t) & \textit{if } c = -1. \end{cases}$$

The support function of a convex body

The support function $s(u)$ of $\mathcal{C}$ is the real map defined on $\mathbb{S}^{n-1}$ by

$$s(u) = \sup_{p \in \mathcal{C}} \langle u, p \rangle.$$

We have the following relation between the width and the support function:

$$W(u) = s(u) + s(-u).$$

Parametrization of the boundary by the support function

Assume $\mathcal{C}$ is strictly convex (it is always the case when it has constant width). Then

Proposition

The image of the map

$$X: \begin{array}{ccc} \mathbb{S}^{n-1} & \to & \mathbb{R}^{n+1} \\ u & \mapsto & s(u)u + \nabla s(u) \end{array}$$

is the boundary of $\mathcal{C}$.

Parametrization of the boundary by the support function

Assume $\mathcal{C}$ is strictly convex (it is always the case when it has constant width). Then

Proposition

The image of the map

$$X : \begin{array}{ccc} \mathbb{S}^{n-1} & \to & \mathbb{R}^{n+1} \\ u & \mapsto & s(u)u + \nabla s(u) \end{array}$$

is the boundary of $\mathcal{C}$*. Moreover, at regular points,* u *is the unit normal of* $\partial\mathcal{C}$*.*

Introducing the averaged support function

We set

$$W/2 := \frac{\int_{\mathbb{S}^{n-1}} s(u)dA}{V(\mathbb{S}^n)},$$

and consider the zero mean map

$$h := s - W/2.$$

Proposition

$\mathcal{C}$ has constant width if and only if $h(u) + h(-u) = 0$.

Corollary

If $\mathcal{C}$ has constant width W, its "fattening" $\mathcal{C}_\epsilon$ has c. w. $W + 2\epsilon$ while its "slimming" $\mathcal{C}_{-\epsilon}$ (if it exists) has c. w. $W - 2\epsilon$.

The converse: the case of curves

Take $h \in C^{1,1}(\mathbb{R}/2\pi\mathbb{Z})$, such that

$$h(t+\pi) + h(t) = 0,\ \forall t \in \mathbb{R}/2\pi\mathbb{Z}$$

and

$$W/2 \geq sup_{t \in \mathbb{R}/2\pi\mathbb{Z}} |h''(t) + h(t)|.$$

Then the curve

$$\gamma(t) = (h(t) + W/2)e^{it} + h'(t) i e^{it}$$

parametrizes the boundary of a convex body with constant width W. Moreover, at regular points, e^{it} is the outward unit normal of γ.

Interpretation in terms of symplectic geometry

- The space of oriented lines $L(\mathbb{R}^n)$ of $\mathbb{R}^n$ has a symplectic structure;
- The set of lines which are normal to a hypersurface is a Lagrangian submanifold;
- The set of normal lines to $\partial\mathcal{C}$ can be identified with the gradient graph

$$\{(u, \nabla s(u)), u \in \mathbb{S}^{n-1}\} \subset T\mathbb{S}^{n-1} \simeq L(\mathbb{R}^n).$$

In other words, the support function s is the generating function of the Lagrangian.

The Wirtinger inequality

Theorem

Let $h \in C^{1,1}(\mathbb{S}^{n-1})$ with vanishing mean and dV the volume element on $\mathbb{S}^{n-1}$. Then

$$\mathcal{E}(h) := \int_{\mathbb{S}^{n-1}} \left(\frac{1}{n-1} |\nabla h|^2 - h^2 \right) dV \geq 0,$$

with equality if h is a first eigenfunction of the Laplacian on the sphere $\mathbb{S}^{n-1}$.

The Wirtinger inequality

Theorem

Let $h \in C^{1,1}(\mathbb{S}^{n-1})$ with vanishing mean and dV the volume element on $\mathbb{S}^{n-1}$. Then

$$\mathcal{E}(h) := \int_{\mathbb{S}^{n-1}} \left(\frac{1}{n-1} |\nabla h|^2 - h^2 \right) dV \geq 0,$$

with equality if h is a first eigenfunction of the Laplacian on the sphere $\mathbb{S}^{n-1}$.

Easy proof using spherical harmonics (higher dimensional equivalent of Fourier series).

The Blaschke formula

Theorem

Let $\mathcal{C}$ be a convex body of in $\mathbb{R}^3$ with constant width W and averaged support function h. Then:

$$V(\mathcal{C}) = \frac{\pi W^3}{6} - \frac{W\mathcal{E}(h)}{2},$$

$$A(\partial\mathcal{C}) = \pi W^2 - \mathcal{E}(h).$$

Corollary (Blaschke)

Let $\mathcal{C}$ a convex body of $\mathbb{R}^3$ with constant width W. Then:

$$V(\mathcal{C}) = \frac{WA(\partial\mathcal{C})}{2} - \frac{\pi W^3}{3}.$$

The Blaschke-Lebesgue problem

Among the convex bodies of $\mathbb{R}^n$ of constant width W, which one has least volume? By the scaling invariance it is equivalent to

$$\min_{\mathcal{C} \text{ c.w.}} \frac{V(\mathcal{C})}{V(B_W)}.$$

The solution is known only in the $n = 2$ case!

Theorem (Blaschke-Lebesgue)

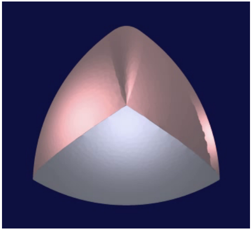
The Reuleaux triangle has less area than any other convex body with the same constant width.

Remark

A crucial step is to prove that the minimizer must be made of arc of circles of same radius.

In dimension 3

The constant width body of $\mathbb{R}^3$ with the best known ratio $V(\mathcal{C})/V(B_W)$ is due to Meissner.



A curvature characterization of the solution

Theorem (A.-Guilfoyle 2010)

Let $\mathcal{C}$ be the solution of the Blaschke-Lebesgue problem in $\mathbb{R}^3$ with constant width W.

Then the smooth parts of its boundary have their smaller principal curvature constant and equal to $1/W$.

Remark

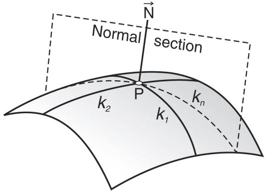
Meissner's body satisfies our criterion.

Remark

The proof uses the support function and some features of dimension $n = 3$.

What are the principal curvatures?

The curvatures k_n of a surface $\mathcal{S}$ at point P are the curvatures of its normal sections:



Definition

- *The principal curvatures of* $\mathcal{S}$ *are* $k_1 := \min k_n$ *and* $k_2 := \max k_n$*;*
- *The mean curvature of* $\mathcal{S}$ *is* $H := \frac{k_1+k_2}{2} = \frac{1}{\pi}\int k_n$*;*

Alternative definition: the principal curvatures of $\mathcal{S}$ at P are the eigenvalues of the shape operator $-d\vec{N}_P \in L(\vec{N}^{\perp}, \vec{N}^{\perp})$.

Back to Blaschke-Lebesgue

Remembering that

$$V(\mathcal{C}) = \frac{\pi W^3}{6} - \frac{W\mathcal{E}(h)}{2},$$

we have

$$\frac{V(\mathcal{C})}{V(B_W)} = 1 - \frac{3\mathcal{E}(h)}{\pi W^2}.$$

Corollary

The ball B_W maximizes the volume among convex bodies with constant width W.

Remark

In order to minimize the volume, one may first decrease W until its smallest possible value $W_0(h)$.

Back to Blaschke-Lebesgue

We prove that

$$W_0(h) = || -\alpha + \sqrt{\alpha^2 - \beta}||_{L^\infty(\mathbb{S}^2)},$$

where we set

$$\alpha := 2h + \Delta h \quad \text{and} \quad \beta := h^2 + h\Delta h + Hess(h).$$

To conclude, a stability argument implies the constancy of

$$k_1 = \frac{W + \alpha + \sqrt{\alpha^2 - 4\beta}}{W^2/2 + \alpha W + \beta}.$$

A generalization

Theorem (A.,2015)

Let $\mathcal{C}$ be a local minimizer of the Blaschke-Lebesgue problem in $\mathbb{Q}^n_c$ with constant width W.

Then the smooth parts of its boundary have their smaller principal curvature constant and equal to $1/\mathcal{T}_c(W)$, where

$$\mathcal{T}_c(t) = \begin{cases} \tan(t) & \text{if } c = 1; \\ t & \text{if } c = 0; \\ \tanh(t) & \text{if } c = -1. \end{cases}$$

The first variation of volume

Let $\mathcal{S}$ an embedded hypersurface and $f \in C^1(\mathcal{S})$. There exists $X \in \mathcal{X}(\mathcal{M})$ such that $X|_{\mathcal{S}} = fN$.

Theorem (First variation formula)

$$\frac{d}{dt} V\big(\exp_{tX}(\mathcal{S})\big)\Big|_{t=0} = -n \int_{\mathcal{S}} H(p) f(p) dV.$$

If, in addition $\mathcal{S} = \partial\mathcal{C}$, then

$$\frac{d}{dt} V\big(\exp_{tX}(\mathcal{C})\big)\Big|_{t=0} = \int_{\mathcal{S}} f(p) dV.$$

Constrained minimization problems

Corollary

- *If $\mathcal{S}$ minimizes volume (or is a critical point of it) among all its "isotopic" hypersurfaces, we have*

$$H(p) = 0, \ \forall p \in \mathcal{S}.$$

We say that $\mathcal{S}$ is minimal;

- *If $\mathcal{S} = \partial\mathcal{C}$ minimizes $V(\mathcal{S})$ among bodies with fixed $V(\mathcal{C})$, we have*

$$H(p) + \lambda\mathbb{1}(p) = 0, \ \forall p \in \mathcal{S}.$$

We say that $\mathcal{S}$ has constant mean curvature.

Constraining the width

$$\exp_{tX}(\mathcal{C}) \text{ has constant width } \forall t$$

$$\iff$$

$$f(p) + f\big(\sigma(p)\big) = 0 \quad \forall p \in \mathcal{S}.$$

Assume by contradiction that $k_1 > 1/\mathcal{T}_c(W)$ somewhere $\longrightarrow$ construct a deformation f that decrease $V(\mathcal{C})$.